HYPERCOMPLEX WAVES

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ABSTRACT

Even though pure plane waves are the easiest solutions of the wave equation, it is known for a long time since the pioneering work of a mere handful of researchers among which Claeys and Leroy, that complex waves are more general solutions being able to explain certain phenomena (the generation of surface waves, the Schoch-effect, the influence of damping,etc.) much more accurately than pure plane waves do. In this work, we present hypercomplex waves as a yet more general solution of the wave equation. We show that every quantity that describes a generalized plane wave may be considered to be hypercomplex. We formulate a generalized law of Snell-Descartes, the dispersion relation and the reflection coefficient for such hypercomplex waves.

THE INTRODUCTION OF HYPERCOMPLEX WAVES

A general solution of the wave equation is written as

$$\mathbf{N} = Ae^{i(\mathbf{k}\bullet\mathbf{r}-\boldsymbol{w}t)}\mathbf{P} \qquad (1)$$

with angular frequency $\boldsymbol{w}$, amplitude

$$A = A_1 + iA_2 + jA_3 + lA_4 \qquad (2)$$

polarization

$$\mathbf{P} = \mathbf{P}_1 + i\mathbf{P}_2 + j\mathbf{P}_3 + l\mathbf{P}_4 \qquad (3)$$

and wave vector

$$\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2 + j\mathbf{k}_3 + l\mathbf{k}_4 \qquad (4)$$

It can be verified that

$$A = |A|e^{i\boldsymbol{j}}\ ;\ \boldsymbol{j} = \boldsymbol{j}_1 + j\boldsymbol{j}_3 + l\boldsymbol{j}_4 \qquad (5)$$

The numbers

$$i, j, l \qquad (6)$$

are hypercomplex units comparable to the complex unit 'i' that obey the following multiplication rules [11-15]:

(7)

*	1	i	j	l
1	1	i	j	l
i	i	-1	l	-j
j	j	-l	-1	i
l	l	j	-i	-1

Hypercomplex numbers have a magnitude defined as

$$|G| = G^{*}G \tag{8}$$

with

$$G = G_1 + iG_2 + jG_3 + lG_4 \tag{9}$$

$$G^{*} = G_1 - iG_2 - jG_3 - lG_4 \tag{10}$$

Tedious calculations lead to

$$\mathbf{u} = \mathrm{Re}\,\mathbf{N} = |A| e^{-\mathbf{k}_2 \bullet \mathbf{r}} \left(E_1\mathbf{P}_1 - E_2\mathbf{P}_2 - E_3\mathbf{P}_3 - E_4\mathbf{P}_4 \right) \tag{11}$$

with for example

$$\begin{aligned} 2E_1 &= \sin(-\boldsymbol{w}t + \boldsymbol{j}_1 + \mathbf{k}_1 \bullet \mathbf{r})[-\sin(\boldsymbol{j}_3 - \mathbf{k}_3 \bullet \mathbf{r})\sin(\boldsymbol{j}_4 - \mathbf{k}_4 \bullet \mathbf{r}) - \sin(\boldsymbol{j}_3 + \mathbf{k}_3 \bullet \mathbf{r})\sin(\boldsymbol{j}_4 + \mathbf{k}_4 \bullet \mathbf{r})] \\ &+ \cos(-\boldsymbol{w}t + \boldsymbol{j}_1 + \mathbf{k}_1 \bullet \mathbf{r})[\cos(\boldsymbol{j}_3 - \mathbf{k}_3 \bullet \mathbf{r})\cos(\boldsymbol{j}_4 + \mathbf{k}_4 \bullet \mathbf{r}) + \cos(\boldsymbol{j}_3 + \mathbf{k}_3 \bullet \mathbf{r})\cos(\boldsymbol{j}_4 - \mathbf{k}_4 \bullet \mathbf{r})] \\ &+ \sin(-\boldsymbol{w}t + \boldsymbol{j}_1 - \mathbf{k}_1 \bullet \mathbf{r})[-\sin(\boldsymbol{j}_3 - \mathbf{k}_3 \bullet \mathbf{r})\sin(\boldsymbol{j}_4 + \mathbf{k}_4 \bullet \mathbf{r}) + \sin(\boldsymbol{j}_3 + \mathbf{k}_3 \bullet \mathbf{r})\sin(\boldsymbol{j}_4 - \mathbf{k}_4 \bullet \mathbf{r})] \\ &+ \cos(-\boldsymbol{w}t + \boldsymbol{j}_1 - \mathbf{k}_1 \bullet \mathbf{r})[-\cos(\boldsymbol{j}_3 - \mathbf{k}_3 \bullet \mathbf{r})\cos(\boldsymbol{j}_4 - \mathbf{k}_4 \bullet \mathbf{r}) + \cos(\boldsymbol{j}_3 + \mathbf{k}_3 \bullet \mathbf{r})\cos(\boldsymbol{j}_4 + \mathbf{k}_4 \bullet \mathbf{r})] \end{aligned} \tag{12}$$

It is clear that one of the four polarizations in (11) will be equal to zero. However, if we want to write the acoustic displacement as a function of acoustic potentials, then we must retain 4 polarizations in order to keep 3 in the displacement (12).

THE DISPERSION RELATION

In order for (1) to be a solution of the wave equation, the dispersion relation must hold

$$\mathbf{k} \bullet \mathbf{k} = \left(\frac{\boldsymbol{w}}{v} + i\boldsymbol{a}_0 \right)^2 \tag{13}$$

with ù the angular frequency, v the sound velocity and $\boldsymbol{a}_0$ the intrinsic damping coefficient of the media. Hence, for

$$\mathbf{k}_2 = \mathbf{á} - \mathbf{â} \tag{14}$$

with á the damping coefficient and â the inhomogeneity as described in numerous texts on inhomogeneous waves [1-11], the following dispersion relations hold:

$$k_1^2 - \boldsymbol{a}^2 - \boldsymbol{b}^2 - k_3^2 - k_4^2 = \frac{\boldsymbol{w}^2}{v^2} - \boldsymbol{a}_0^2 \tag{15}$$

$$\mathbf{k}_1 \bullet \mathbf{k}_2 = k_1\boldsymbol{a} \tag{16}$$

$$\mathbf{k}_3 \perp \mathbf{k}_1 \tag{17}$$

$$\mathbf{k}_4 \perp \mathbf{k}_1 \tag{18}$$

GRAPHICAL REPRESENTATION OF HYPERCOMPLEX WAVES

In Figure 1 and Figure 2, some examples of hypercomplex waves propagating in the y-direction are given. The parameters used are shown in Table 1:

	k_1	$\boldsymbol{a}$	$\boldsymbol{b}$	k_3	k_4	comments
Fig 1a	3	0	0.1	0	0	≈ undamped inhomogeneous wave
Fig 1b	3	0.3	0.1	0	0	≈ damped inhomogeneous wave
Fig 1c	3	0	0	1	0	
Fig 1d	3	0	0	1	1	
Fig 2	3	0.3	0.1	1	1	

Table 1: The parameters that are applied to produce figure 1 and figure 2

Pure plane waves have an equiphase surface directed normal to the propagation direction. In that equiphase surface, the amplitude is the same everywhere. Damped plane waves are almost the same as pure plane waves except for the fact that the amplitude diminishes exponentially with a coefficient á in the direction of propagation.

If damping is concerned, a discontinuity interface between two media almost naturally induces the generation of inhomogeneous waves, with an equiphase surface normal to the direction of propagation, but with an amplitude varying exponentially with a coefficient â in that equiphase surface. The dispersion relation also allows inhomogeneous waves in the absence of damping.
Hypercomplex waves are an extension of the inhomogeneous waves in that the variation of the amplitude inside the equiphase plane is not limited to exponential growth or decay, but can also be harmonic in space. Moreover, the 'equiphase plane' that exists for inhomogeneous waves can be replaced, due to the particular values of k_3 and k_4, by a 'generalized equiphase plane' in which the phase switches from a reference phase î to a phase î-ð periodically in space along the direction normal to the direction of propagation. Hence, the hypercomplex waves have much more degrees of freedom than the complex waves, whence it is plausible that some physical phenomena that are still unexplained using the existing wave models will be understood better in future exploiting the more general features of hypercomplex waves.

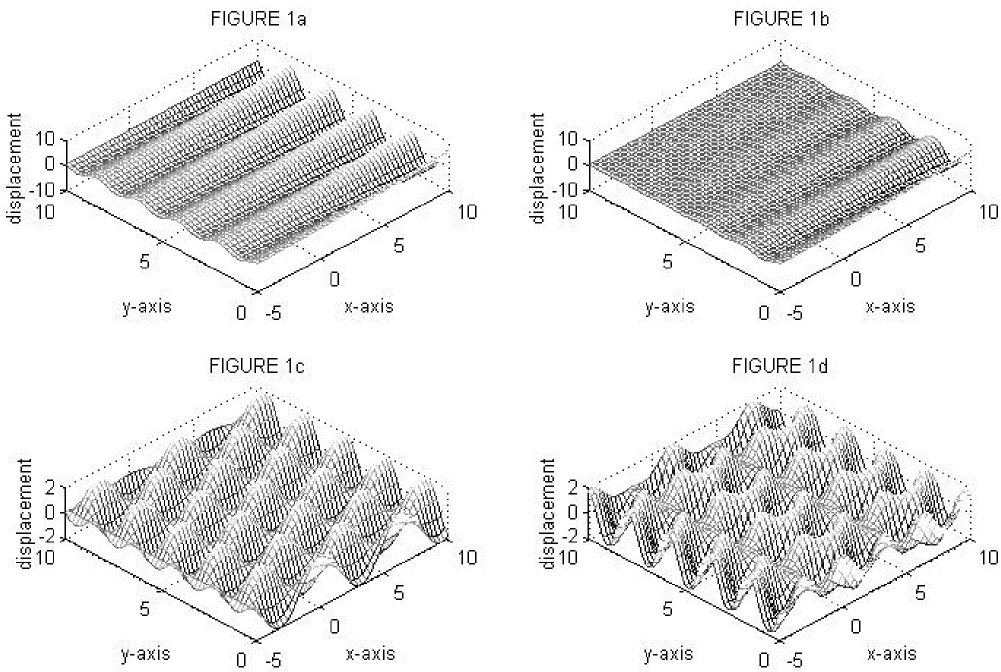

Figure 1: Some examples of hypercomplex waves.
The different parameters are shown in table1

THE GENERALIZED LAW OF SNELL-DESCARTES

If we take into account the properties derived using the dispersion relation, we may write

$$k = \begin{pmatrix} {}^{(1)}k \\ {}^{(2)}k \\ {}^{(3)}k \end{pmatrix} = \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} {}^{(1)}k_2 \\ {}^{(2)}k_2 \\ {}^{(3)}k_2 \end{pmatrix} + j \begin{pmatrix} 0 \\ {}^{(2)}k_3 \\ {}^{(3)}k_3 \end{pmatrix} + l \begin{pmatrix} 0 \\ {}^{(2)}k_4 \\ {}^{(3)}k_4 \end{pmatrix} \qquad (19)$$

using the base

$$\begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_{\|} \\ \mathbf{e}_{\perp} \end{pmatrix} \qquad (20)$$

$\mathbf{e}_w$ being a unit vector along the direction of propagation of the wave, $\mathbf{e}_{\|}$ a unit vector parallel to the interface with $\mathbf{e}_{\|} \perp \mathbf{e}_w$ and $\mathbf{e}_{\perp}$ a unit vector with $\mathbf{e}_{\perp} \perp \mathbf{e}_{\|}$ and $\mathbf{e}_{\perp} \perp \mathbf{e}_w$. We define the orientation as

$$\mathbf{e}_{\parallel} \times \mathbf{e}_{\perp} = \mathbf{e}_w \tag{21}$$

and

$$\mathbf{e}_{\perp} \bullet \mathbf{n} \leq 0 \tag{22}$$

with **n** a normal vector on the interface, pointing to the incidence media. We denote the incoming wave by 'inc' and the refracted waves by 'p'. The angle J^p is defined by

$$\begin{pmatrix} \mathbf{e}_w^p \\ \mathbf{e}_{\parallel}^p \\ \mathbf{e}_{\perp}^p \end{pmatrix} = \begin{pmatrix} \sin J^p & 0 & \cos J^p \\ 0 & 1 & 0 \\ -\cos J^p & 0 & \sin J^p \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} \tag{23}$$

The angle J^{inc} is defined by (23) for '*p*' replaced by '*inc*'.
The generalized law of Snell-Descartes becomes

$$w^{inc} = w^p \tag{24}$$

$$\mathbf{k}^{inc} \bullet \mathbf{r} = \mathbf{k}^p \bullet \mathbf{r} \tag{25}$$

for **r** lying on the interface. When the dispersion relation is taken into account, (25) yields

$$^{(2)}\mathbf{k}_2^{inc} = {}^{(2)}\mathbf{k}_2^p \tag{26}$$

$$^{(2)}\mathbf{k}_3^{inc} = {}^{(2)}\mathbf{k}_3^p \tag{27}$$

$$^{(2)}\mathbf{k}_4^{inc} = {}^{(2)}\mathbf{k}_4^p \tag{28}$$

$$\sin J^{inc}\, {}^{(1)}\mathbf{k}_1^{inc} = \sin J^{(1)}\mathbf{k}_1^{p\,p} \tag{29}$$

$$\sin J^{inc}\, {}^{(1)}\mathbf{k}_2^{inc} - \cos J^{inc}\, {}^{(3)}\mathbf{k}_2^{inc} = \sin J^{p}\, {}^{(1)}\mathbf{k}_2^{p} - \cos J^{p}\, {}^{(3)}\mathbf{k}_2^{p} \tag{30}$$

$$\cos J^{inc}\, {}^{(3)}\mathbf{k}_3^{inc} = \cos J^{p}\, {}^{(3)}\mathbf{k}_3^{p} \tag{31}$$

$$\cos J^{inc}\, {}^{(3)}\mathbf{k}_4^{inc} = \cos J^{p}\, {}^{(3)}\mathbf{k}_4^{p} \tag{32}$$

It is easily verified that (26-32) are a generalization of the generalized Snell-Descartes law for complex waves found in numerous articles such as [11].

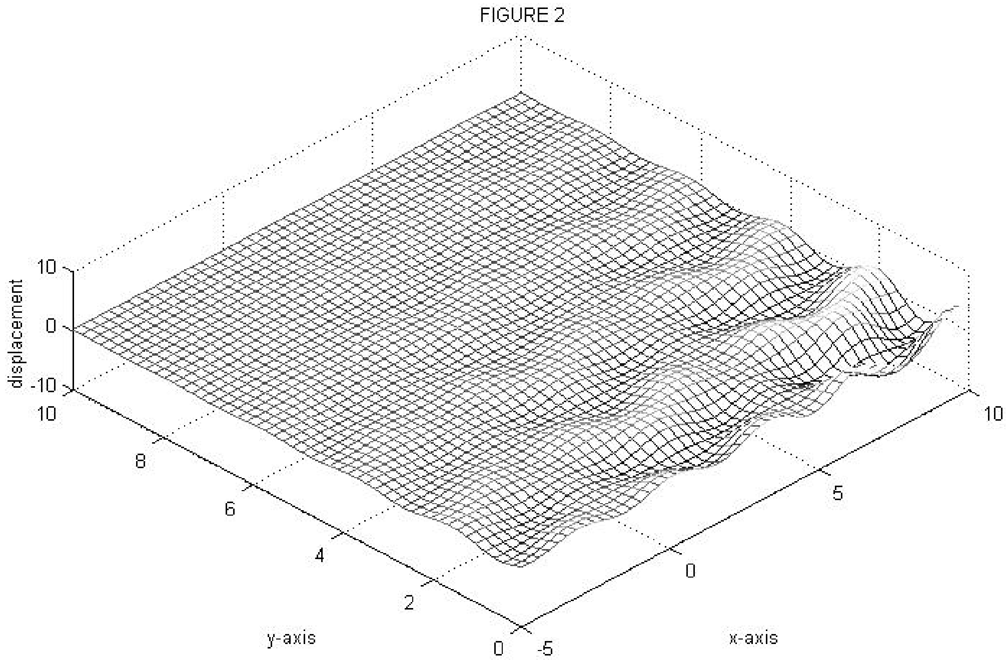

Figure 2: Another example of a hypercomplex wave. The parameters are shown in table1

CALCULATION OF THE REFLECTION COEFICIENT

The calculation of the reflection coefficient for an interface between two isotropic media cannot be described in just a few pages. Therefore we shall limit ourselves to the main features of hypercomplex waves that must be taken into account. Most computer languages are not suitable to calculate in hypercomplex space as it is. Therefore it is necessary to expand all values and perform all algebraic calculations manually before programming. The main concern during the algebraic calculations is the fact that hypercomplex numbers do not commute. The main consequence of this is that inversion of the continuity matrix shall be limited to the right inverse (as opposed to the left inverse). Just as for complex waves, the continuity conditions must hold for (normal) stress and (normal) displacement. For an incidence direction in the xz-plane, the y-component of the incoming wave can be separated from the xz-components, whence we get a continuity system of equations of the form

$$\left(y^{inc} P_y^{inc,s}\right) E \times RIB = \left(y^{1} P_y^{1,s} \quad y^{2} P_y^{2,s}\right) \tag{33}$$

$$\left(j^{inc} \quad y^{inc} PSV^{inc,s}\right) I \times RIA = \left(j^{1} \quad y^{1} PSV^{1,s} \quad j^{2} \quad y^{2} PSV^{2,s}\right) \tag{34}$$

in which $\times$ is the matrix product, *E* and *I* are matrices, *RIA* is the right inverse of a matrix *A* and *RIB* is the right inverse of a matrix *B*.

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