CONSTRUCTION OF ANALYTICAL SOLUTIONS FOR THE ERROR ESTIMATION OF ACOUSTICAL BOUNDARY ELEMENT SOLVERS

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ABSTRACT

The boundary element method is an efficient tool for the calculation of sound radiation from complex structures. However, exact analytical solutions are needed, in order to determine the real error of boundary element calculations. Such analytical solutions are generally not available for radiating bodies with arbitrary surface geometry. But, if the body is forced to vibrate with the corresponding normal surface velocity of a multipole, the radiated sound field is analytically known and can be used for determining the exact error of the boundary element method. Such multipole errors of increasing order will be investigated for a sphere and a non-convex cat's-eye structure.

INTRODUCTION

The sound radiated from complex shaped vibrating structures can be calculated by means of the boundary element method (BEM). For estimating the accuracy of such calculations exact analytical solutions are needed. Unfortunately, such analytical solutions are only available for radiators of very simple shape, e. g. for spheres. However, exact solutions can be constructed for closed vibrating surfaces of arbitrarily complex shapes, if the normal velocity field on the surface is chosen in a special way. For that reason, the normal velocity of a multipole located in the interior of the radiator is calculated on the boundary, which is assumed to be sound-transparent for the moment. Then this special velocity is introduced into the boundary element program.
Now, the sound pressure on the surface and in the exterior space can be calculated and directly compared with the known solution of a multipole. For a monopole this procedure is known as One-Point-Source-test (OPS-test).
However, calculations with the an iterative variant of the BEM show that the OPS-test may lead to small errors, whereas multipoles of higher order generate larger errors. This demonstrates that the error of the BEM depends both on the geometry of the structure and on the spatial distribution of the normal velocity on the surface of the radiator.

THE MULTIPOLE TEST

Multipoles, also called radiating spherical wave functions, are solutions of the Helmholtz equation in spherical coordinates in the infinite, three-dimensional space. They are given by [1]

$$\boldsymbol{y}_{nm}^{c,s}(x) = \Gamma_{nm} h_n^{(2)}(kr) P_n^m(\cos \boldsymbol{J}) \cdot \begin{cases} \cos m\boldsymbol{j} \\ \sin m\boldsymbol{j} \end{cases} \tag{1}$$

with the associated Legendre polynomials P_n^m. Spherical coordinates $x = (r\sin J \cos\boldsymbol{j}, r\sin J \sin\boldsymbol{j}, r\cos J)^T$ are used, where *T* denotes transposition. The index c or s denotes the Cosine or Sine, and *k* is the wave number. The cylinder functions $h_n^{(2)}$ are the spherical Hankel functions of the second kind. The normalization factors Γ_{nm} are chosen such, that the spherical harmonics

$$y_{nm}^{c,s}(x) = \Gamma_{nm} P_n^m(\cos \boldsymbol{J}) \cdot \begin{cases} \cos m\boldsymbol{j} \\ \sin m\boldsymbol{j} \end{cases} \tag{2}$$

are orthogonal with respect to the integration over the unit sphere [1].
If a structure is vibrating with a certain normal surface velocity *v*, then the corresponding sound field radiated into the exterior space is uniquely determined. Hence, if the normal velocity of a multipole (*n* = exterior normal)

$$v^{n,m,l} = \frac{j}{\boldsymbol{wr}} \frac{\partial \boldsymbol{y}_{nm}^{c,s}}{\partial n} \tag{3}$$

is enforced on the structural surface *S*, the corresponding sound pressure $p_w^{n,m,l} = y_{nm}^{c,s}$ of Eq. (1) will result (*w* = true, *l* = c or s, ù = circular frequency, and ñ = density of the fluid).
If this normal surface velocity (3) is introduced as boundary condition and input into a computer program like for example a boundary element solver or a source simulation technique as the full-field method [2], the program gives the sound pressure $p_{sim}^{n,m,l}$ (*sim* = simulated), which will deviate from the true sound pressure due to the numerical inaccuracies of the method. For iterative BE-solvers [3-5], we can define errors on different levels of the calculation.

KINDS OF MULTIPOLE ERRORS

The starting point of the BEM is the Helmholtz integral equation

$$p(x) = 2\iint_S p(y) \frac{\partial}{\partial n_y} g(x,y) ds_y + 2\iint_S j\boldsymbol{wr} v g(x,y) ds_y \tag{4}$$

(*g* is the Green´s function) or in evidently abbreviated notation

$$p = Lp + f(v) \tag{5}$$

for the calculation of the pressure *p* on the surface of the radiator *S* which is assumed to be smooth enough [6].
The multipole error of the first kind

$$E_1^{n,m,l}(x) = \left| p_w^{n,m,l}(x) - p_{disc}^{n,m,l}(x) \right| / N \quad \text{for} \quad x \in S \tag{6}$$

with

$$N = \max\left[\max_{\{S\}} \left| p_w^{n,m,l}(x) \right|, \max_{\{S\}} \left| p_{disc}^{n,m,l}(x) \right| \right]$$

is obtained by inserting the known multipole solution $p_w^{n,m,l}$ into the Helmholtz operator

$$p_{disc}^{n,m,l} = Lp_w^{n,m,l} - f(v^{n,m,l}) \tag{7}$$

and comparing the resulting sound pressure $p_{disc}^{n,m,l}$ (*disc* = discretization) with the analytical solution.

Expression (6) is a local error, which can differ from surface element to surface element. For obtaining an integral relative error, the corresponding error amplitude will be squared, integrated over the entire surface, and normalized:

$$ES_1^{n,m,l} = \frac{\sqrt{\iint_S \left|p_w^{n,m,l}(x) - p_{disc}^{n,m,l}(x)\right|^2 ds}}{\max\left[\sqrt{\iint_S \left|p_w^{n,m,l}(x)\right|^2 ds}, \sqrt{\iint_S \left|p_{disc}^{n,m,l}(x)\right|^2 ds}\right]} \quad . \tag{8}$$

Obviously, this error is a measure for the quality of the approximation of the Helmholtz operator, i. e. for the accuracy of the numerical integration and the discretization of the structure. Thus, this error will be called ***m**ultipole **d**iscretization **e**rror* (MPDE) in the following. It should be taken into account, that the MPDE does not depend on the procedure how the system of equations (5) is solved (e. g., directly or iteratively).
For estimating the total error of the BE-method, a multipole error of second kind will be defined as follows: The system of equations (5)

$$p_{BE} = Lp_{BE} + f\left(v^{n,m,l}\right) \tag{9}$$

is solved directly or iteratively for the prescribed normal multipole velocity $v^{n,m,l}$, which gives a more or less good numerical estimation for the sound pressure p_{BE}. In analogy to the above procedure, a local error

$$E_2^{n,m,l}(x) = \left|p_w^{n,m,l}(x) - p_{BE}(x)\right| / N \tag{10}$$

or an normalized integral error

$$ES_2^{n,m,l} = \frac{\sqrt{\iint_S \left|p_w^{n,m,l}(x) - p_{BE}(x)\right|^2 ds}}{\max\left[\sqrt{\iint_S \left|p_w^{n,m,l}(x)\right|^2 ds}, \sqrt{\iint_S \left|p_{BE}(x)\right|^2 ds}\right]} \quad . \tag{11}$$

can be defined. The constant N is defined in analogy to the one in Eq. (6). This kind of error depends both on the discretization and on the (numerical) variant of the BE method, and hence it is called ***m**ultipole **m**ethod **e**rror* (MPME) For example, the MPME changes from iteration step to iteration step, if an iterative BE-solver is used. If the iterative method converges against the true solution, the MPME decreases with the number of iterations until the discretization error MPDE is reached in the ideal case. Furthermore, both kinds of multipole errors depend on the choice of the location of the multipole (i.e. the singularity of the source function). The formulation given in Eq. (1) assumes silently that the multipole is located in the origin of the coordinate system. Of course this is not necessary and often not possible as for the cat's-eye (see next chapter). But it is required that the singularity of the source lies inside the radiating structure.
In some publications the so-called "One-point source test" (OPS test) is also mentioned [7]. This leads to the multipole error of second kind of the order $(n, m, l) = (0,0,0)$. Therefore, it is a monopole error and was recently used for investigating the quality of BE methods in [8].

NUMERICAL RESULTS

First, the multipole error for a vibrating sphere in air with 1 *m* radius at a frequency of 100 Hz will be calculated for increasing multipole orders with the help of the iterative BE method described in [5]. The discretized surface of the sphere consists of about 6000 elements (see Fig. 2). The calculations show that the MPME (11) increases from 0.3 % at order $n = m = 6$ to almost 4 % at $n = m = 20$. In Fig. 1 the directivity pattern of the sound pressure level is represented in the far-field region in 100 m of distance in the *xy* plane around the sphere.
The circles show the exact sound pressure level, the solid line represents the second iteration of the BE solver. One can see greater deviations at two neighbouring maxima - altogether four times. The local MPDE (6) also reflects this behaviour as shown in Fig. 2: one recognizes two of the four coupled error maxima on the surface of the sphere. The amplification of the surface error

from only 0.3 % up to several decibels in the far-field region (see Fig. 1) can be explained as follows: The used multipole of order (6,6,0) mainly generates a pure near-field at 100 Hz, which hardly radiates into the far-field. Small errors (of monopole or dipole character, for example) may emit much more effectively. At 1000 Hz the error disappears almost completely due to the higher radiation efficiency of the (6,6,0) multipole at higher frequencies.

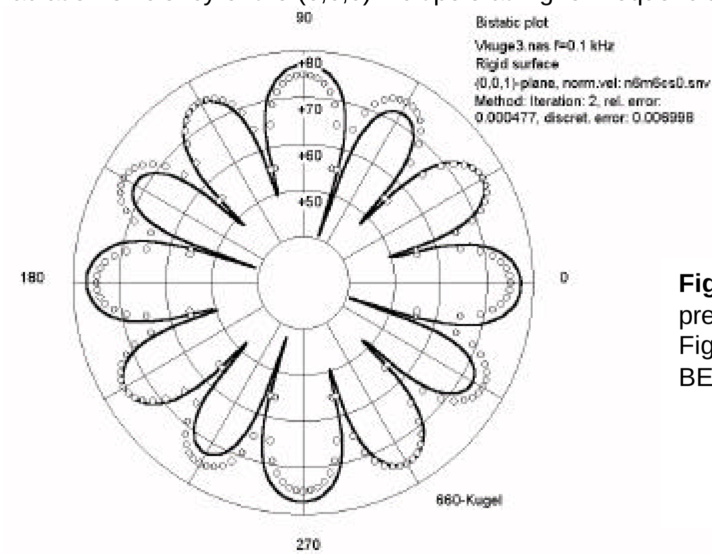

Fig.1: Directivity pattern of the sound pressure level for the sphere shown in Fig. 2, *f* = 100 Hz; solid line = iterative BE solution, circles = analytical solution.

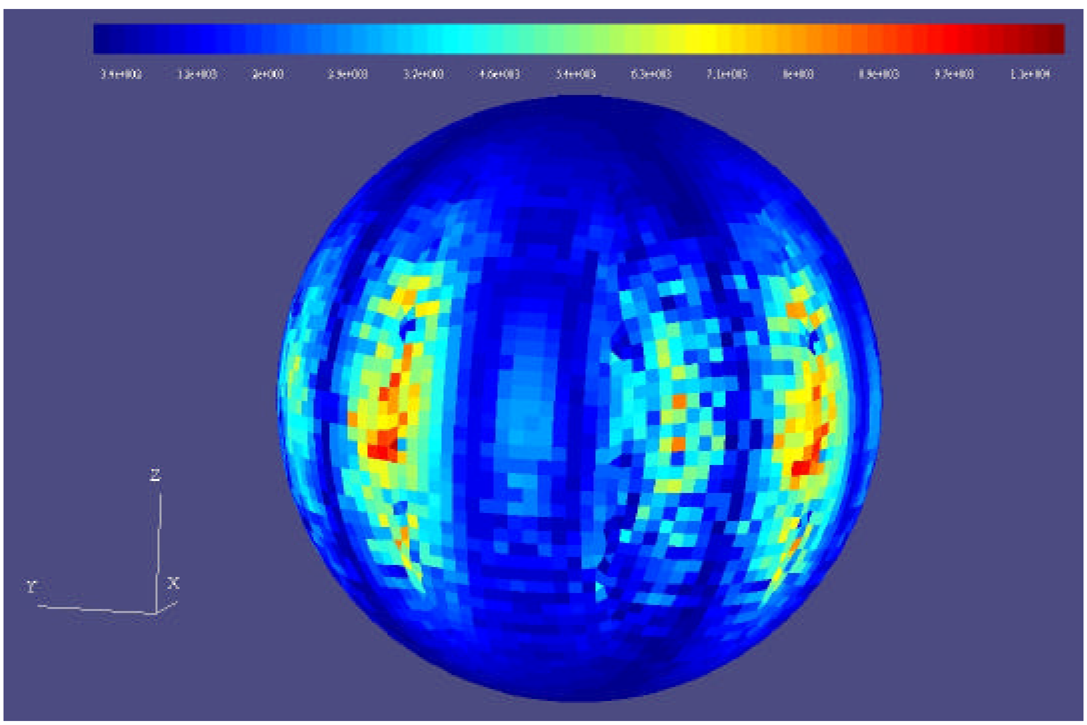

Fig. 2: MPDE (6) of order $n = m = 6$ on the surface of the sphere at *f* = 100 Hz.

Second, a monopole is placed in a non-convex structure near the origin at $(-0.2m, -0.2m, -0.2m)$. In [2,3,6], the scattering from such a structure was studied which consists of a sphere where the positive octant, i. e. the part corresponding to $x > 0, y > 0, z > 0$, was cut out. The corresponding radius of the cat's-eye is 1*m*. It consists of 7911 boundary elements. The structure is called "cat's-eye", since it acts like a three-dimensional reflector. The monopole radiates at a frequency of 500 Hz. As shown in Fig. 3, the directivity pattern of the sixth iteration (solid line) and the analytical solution (circles) agree very well in the far-field. The discretization error MPDE (8) is 2.4 %. However, the total error MPME (11) is 10.4% for the sixth iteration.

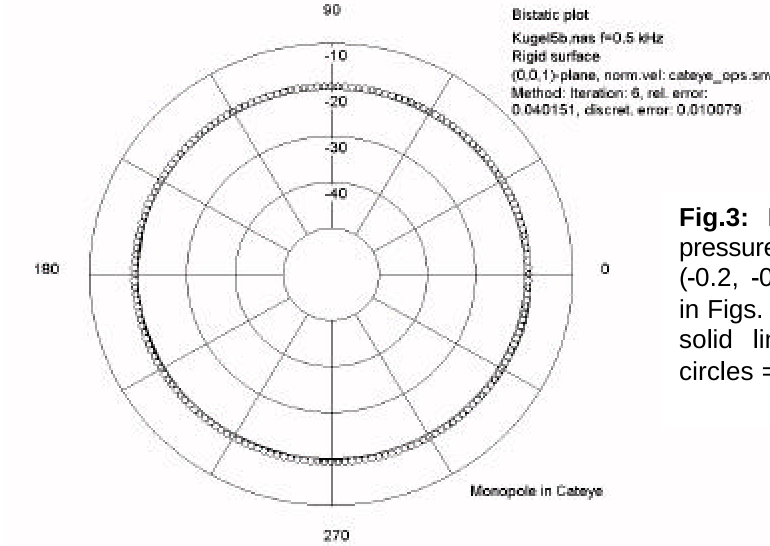

Fig.3: Directivity pattern of the sound pressure level around the point (-0.2, -0.2, -0.2) for the cat's-eye shown in Figs. 4 and 5; *f* = 500 Hz;
solid line = 6th iterative BE solution, circles = analytical solution.

In Fig. 4, the corresponding local error (10) is shown. It can be seen that the largest errors occur at the origin.

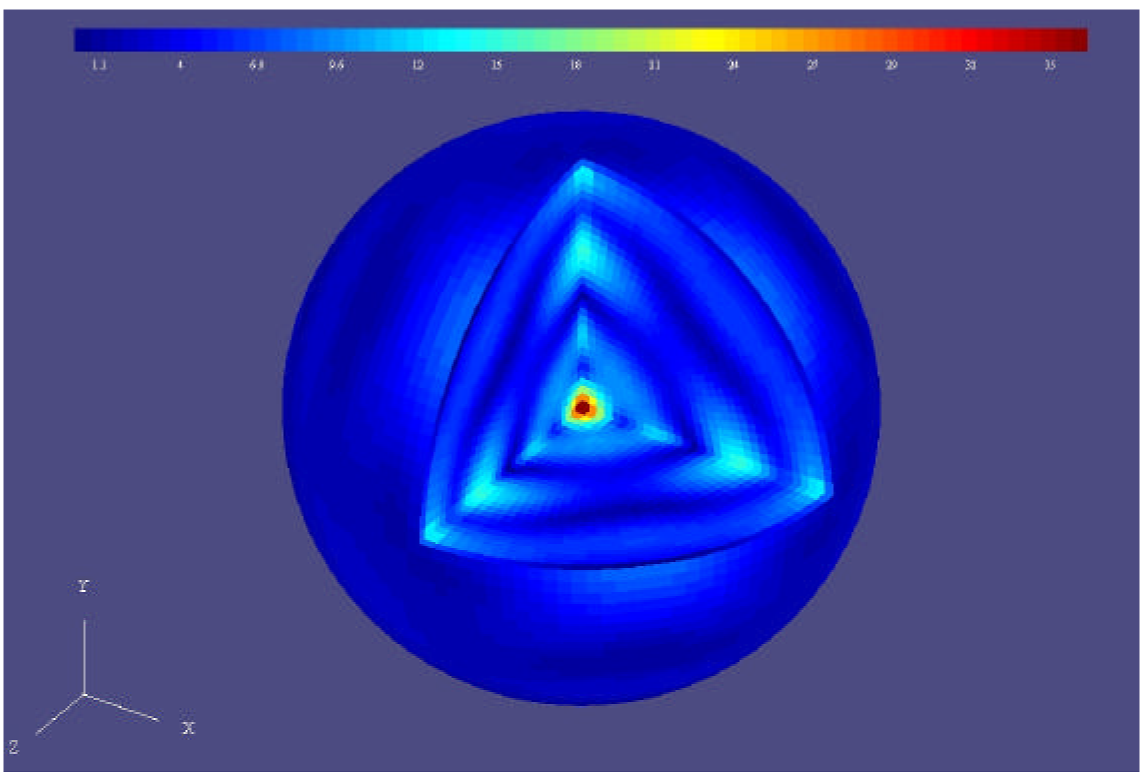

Fig. 4: Local error of order $n = 0, m = 0$ on the surface of the cat's-eye at *f* = 500 Hz.

In Fig. 5, the local error (10) is divided by $N(x) = \max\left[\left|p_w^{n,m,l}(x)\right|, \left|p_{BE}(x)\right|\right]$ instead of *N* (see Eq. (10). Clearly , this relative error is bigger for points with small pressure values. Now, the maxima of the error occur at the three corners of the positive octant.

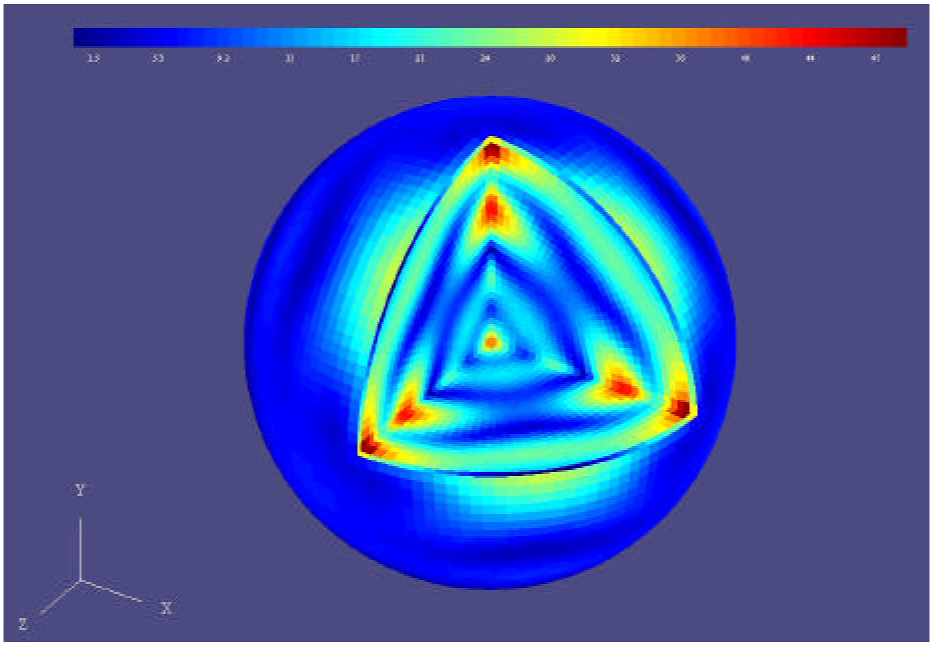

Fig. 5: Relative, local error of order $n = 0, m = 0$ on the surface of the cat's-eye at f = 500 Hz.

CONCLUSION AND OUTLOOK

Which benefit do the calculated multipole errors have for a practical radiation or scattering calculation task ? In many applications the sound radiation from a vibrating structure must be computed for a given surface velocity v. If one selects a multipole velocity which oscillates on the surface more strongly than v, then it can be suspected that the accompanying multipole error MPME represents an upper limit for the real error. For determining the real error exactly, the velocity v has to be expanded into a series of multipole velocities. This is possibly in principle but generally too expensive, since such a development corresponds to a complete calculation by means of the source simulation technique [1]. A detailed analysis of the errors MPDE and MPME for various and more complex structures is carried out at present.

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