

ACOUSTICS OF THE LAGRANGIAN-AVERAGED 1D COMPRESSIBLE EULER EQUATIONS

J. I. Ramos¹

¹Escuela de Ingenierías, Universidad de Málaga, Dr. Ortiz Ramos, s/n, 29071 Málaga, Spain
{jirs@lcc.uma.es}

Abstract

Starting from a one-dimensional Lagrangian-averaged Euler compressible flow model that includes the Euler equations when an averaging or coarsening parameter is nil, we develop leading-order asymptotic equations as functions of the flow Mach number and the averaging/coarseness parameter, and show that the acoustic pressure may be governed by the standard second-order wave equation with additional terms which may include second- or fourth-order spatial derivatives. We also discuss the physical significance of the Lagrangian-averaged model and prove that, under certain conditions, such a model is governed at leading-order by Love's equation which has also been used to study acoustic wave propagation in bubbly liquids and elastic phenomena. A simple modification of the coarseness terms in the Lagrangian-averaged compressible flow model yields the Stokes' acoustic wave equation which was previously derived from the one-dimensional Navier-Stokes equations and that also governs heat and mass transfer with delays between the fluxes and the flow gradients, seepage flows, etc. This latter equation has been solved by means of a second-order accurate, time-linearized, semi-implicit finite difference method and the results show that an initial pressure condition may evolve into a steep front and a dissipative solitary wave (soliton) in infinite domains.

Keywords: acoustic wave equations, Lagrangian-averaged Euler models, perturbation methods, Love's equation, Stokes' acoustic equation

PACS nos. 43.25.+y, 47.40.-x, 47.35.Rs

1 Introduction

Lagrangian-averaged flow models were introduced to model the effects of fluctuations on the mean flow [1,2]. In these models, averaging is performed at the level of variational principles rather than on the governing Euler or Navier-Stokes equations. Lagrangian-averaged incompressible fluid models for the Euler and Navier-Stokes equations are referred to as Lagrangian-averaged Euler (LAE- α) and Lagrangian-averaged Navier-Stokes (LANS- α) equations, respectively, and the study of these models has shed some light onto the well-posedness and the regularity of the solutions of the three-dimensional Euler and Navier-Stokes equations [3,4]. In addition, some Lagrangian-averaged incompressible fluid models have been successfully used as turbulence models. For compressible fluids, Lagrangian-averaged models are usually governed by lengthy equations that may involve high-order derivatives and inverse elliptic operators, especially in two and three dimensions [5].

Bhat and Fetecau [6] developed a one-dimensional, Lagrangian-averaged compressible fluid model for barotropic fluids based on a modified Lagrangian that includes not only the kinetic and potential

energies but also the semi-positive term $\alpha^2 \rho u_x^2/2$, where α is a small positive number that represents the coarseness of the average. The minimization of the action provides the following equations

$$\rho_t + (\rho u)_x = 0, \quad (1)$$

$$\rho(u_t + u u_x) = -p_x + \alpha^2 S(u), \quad (2)$$

where

$$S(u) = \rho_x u_{xt} + \rho_x u u_{xx} + \rho u_x u_{xx} + \rho u_{xxt} + \rho u u_{xxx}, \quad (3)$$

t is time, x is the spatial coordinate, and ρ , p and u are the density, pressure and velocity, respectively, and α has the dimensions of length.

Equations (1)-(3) become the well-known one-dimensional Euler's equations for $\alpha = 0$ that only contain first-order derivatives with respect to time (t) and space (x). Note that Eq. (3) contains third-order derivatives with respect to space.

Equations (1)-(3) have been the subject of study by Keiffer et al. [7] who considered the propagation of weakly-nonlinear, dissipative acoustic solitons in single-phase lossless fluids by assuming that the acoustic pressure, i.e., the pressure difference between the actual pressure and the equilibrium one, is a quadratic function of the acoustic density and showed that, if regularization terms are included at leading-order in the asymptotic expansions in power series of the Mach number, the resulting acoustic equation is the well-known (inviscid) weakly nonlinear acoustic wave equation previously derived by Lesser and Seebass from the compressible Euler equation [8]. Although, their formulation yielded the Love equation of classical elasticity theory [9] for zero Mach numbers and the inviscid Blackstock-Crighton-Lesser-Seebass or weakly-nonlinear acoustic wave equation for $\alpha = 0$ [8,10], their derivation was based on the assumption that the nondimensional acoustic density is on the order of the Mach number times the derivative of the velocity potential with respect to time. Such an approximation has frequently been used in the acoustic literature for the Euler equations [11-13] but not for the Lagrangian-averaged Euler Eqs. (1)-(3); thus, its application is only an approximation.

In this paper, we consider the Lagrangian-averaged Euler Eqs. (1)-(3) and first show that Eq. (2) can be written in a simple conservation-law form after some easy but lengthy algebra. We then combine the Euler and Lagrangian-averaged Euler equations to obtain some linear wave equations that contain second- and fourth-order spatial derivatives, and then present some modified Lagrangian-averaged models that result in other acoustic equations. Finally, a generalized model that includes nonlinearities, damping and the Stokes' and Love's terms is presented and some numerical results are illustrated. Unless otherwise stated, we shall limit our presentation to one-dimensional motions of barotropic fluids in the absence of body forces.

2 Acoustic wave equations

After some tedious algebra, it may be shown that Eq. (2) can be written in conservation form as

$$L(\rho u) = -p_x + \alpha^2 [L(\rho u_x)]_x, \quad (4)$$

where $L(\rho \theta) = (\rho \theta)_t + (\rho u \theta)_x$, and Eq. (1) can also be written as $L(\rho) = 0$. Moreover, partial differentiation of Eq. (2) with $\alpha = 0$ with respect to x , i.e., partial differentiation of the Euler equation with respect to x , yields, after some rearrangements

$$L(\rho u_x) = -\rho (u_x)^2 - \rho (p_x/\rho)_x, \quad (5)$$

which can be substituted into Eq. (4) to obtain

$$\rho(u_t + u u_x) = -p_x - \alpha^2 [\rho (u_x)^2 + \rho (p_x/\rho)_x], \quad (6)$$

which together with the continuity equation and the equation of state for a barotropic fluid, i.e., $p = f(\rho)$, provide three equations for the three unknowns ρ , p and u . The substitution of Eq. (5) into Eq. (4) is justified provided that α is small which is the condition imposed on Lagrangian-averaged models. Such a substitution implies that the second term in the right-hand side of Eq. (4) which corresponds to a Lagrangian-averaged Euler compressible fluid model can be replaced by that derived from the standard Euler's equation.

2.1 Mixed Euler-Lagrangian-averaged Euler models: Cut-off frequencies

By assuming that the density and pressure can be written as the sum of their equilibrium and fluctuating (acoustic) values, substitution of these values into Eqs. (1) and (6) as well as into the equation of state for a barotropic fluid, i.e., $p = K \rho^n$, where K is a constant and n is the (constant) barotropic exponent, one can easily obtain the following linear equations after neglecting nonlinear terms

$$\rho'_t + \rho_e u'_x = 0, \quad p' = c_e^2 \rho', \quad (7)$$

$$p'_{tt} - c_e^2 p'_{xx} = \alpha^2 c_e^2 p'_{xxxx}, \quad (8)$$

where the subscript e denotes equilibrium conditions, the primes denote fluctuating (acoustic) values and c is the speed of sound.

Equation (8) reduces to the one-dimensional, linear wave equation for $\alpha^2 = 0$ and contains a fourth-order spatial derivative. A Fourier analysis of Eq. (8), i.e., $p'(t, x) \sim \exp(i(kx - \omega t))$, where $i^2 = -1$, and k and ω denote the wavenumber and angular frequency, respectively, shows that the dispersion relation corresponding to Eq. (8) is

$$\omega^2 = k^2 c_e^2 (1 - \alpha^2 k^2), \quad (9)$$

and, therefore, the frequency is real for wavenumbers such that $1 \geq \alpha^2 k^2$ and complex, otherwise. This means that the formulation presented here is only valid for wavelengths that are larger than or equal to $2\pi\alpha$; therefore, the validity of the model presented here increases as α is decreased. Note that the Lagrangian-averaged Euler model presented here reduces to the Euler equation for $\alpha = 0$. Equation (9) also indicates that the frequency is nil for $k = 0$ and $1/\alpha$.

2.2 Perturbation methods: Love's equation

As stated above, the Lagrangian-averaged Euler compressible fluid model is governed by Eqs. (1) and (4) and the state equation $p = K \rho^n$, where α^2 is a small positive number. The presence of α^2 in the right-hand side of Eq. (4) suggests that ρ , p and u be expanded in asymptotic power series of α^2 as

$$\varphi = \varphi_0 + \alpha^2 \varphi_2 + O(\alpha^4), \quad (10)$$

where $u_0 = 0$. Substitution of Eq. (10) into Eqs. (1) and (4) and the equation of state yields, to $O(\alpha^2)$,

$$(\rho_2)_t + (\rho_0 u_2)_x = 0, \quad p_2 = n K \rho_0^{n-1} \rho_2, \quad (11)$$

$$\rho_0 (u_2)_t = - (p_2)_x + \alpha^2 \rho_0 (u_2)_{tx}, \quad (12)$$

which can be combined to yield

$$(\rho_2)_{tt} - n K (\rho_0)^{n-1} (\rho_2)_{xx} = \alpha^2 (\rho_2)_{xxtt}, \quad (13)$$

with analogous expressions for p_2 and u_2 .

Equation (13) is the well-known Love's equation which governs the longitudinal motion of a slender, elastic rod when the lateral displacements are taken into account and was derived by Love by means of a variational approach [9]. Equation (13) has also been derived by Keiffer et al. [7] as indicated in the Introduction. However, Keiffer et al. [7] assumed that the acoustic pressure is a quadratic expression of the acoustic density, the Mach number is small and the ratio of the acoustic density to the equilibrium density is linearly proportional to the Mach number, and the time derivative of the velocity potential; the latter is a frequently used approximation in acoustics of inviscid media governed by the Euler's equations [11]. By way of contrast, Eq. (13) has been deduced here without making any assumption about the Mach number; the only assumptions made was that the fluid is barotropic and the flow variables can be expanded as asymptotic powers series of the small parameter α^2 that appears in the Lagrangian-averaged Euler model. Equation (13) also appears in the modeling of transient acoustic waves in isothermal bubbly fluids [14,15], in the theory of seepage of homogeneous liquids through fissured rocks [16], etc. It must be pointed out that, in Eq. (13), $n K (\rho_0)^{n-1}$ is equal to c_e^2 .

Compared with Eq. (8), a Fourier analysis of Eq. (13) indicates that

$$\omega^2 = k^2 c_e^2 / (1 + \alpha^2 k^2), \quad (14)$$

and, therefore, the frequency is semipositive; it is only zero for $k = 0$, and behaves as $\omega^2 \approx k^2 c_e^2$ for $1 \gg \alpha^2 k^2$ and $\omega^2 \approx c_e^2 / \alpha^2$ $1 \ll \alpha^2 k^2$; of these two limits, only the first one is in accord with assumption that α^2 must be small in Lagrangian-averaged Euler models.

A comparison between the models presented in the previous section and this one, i.e., between Eqs. (8) and (13), respectively, clearly indicates that their differences are entirely due to the fact that, in the model presented in the previous section, the characteristic term of the Lagrangian-averaged Euler model was replaced by that derived from the Euler's equations, whereas, in the acoustic model presented in this section, only the Lagrangian-averaged Euler equations were used.

2.3 Asymptotic Lagrangian-averaged Euler models: Love's equation

The derivation presented in previous paragraphs of last section was based on dimensional equations; a more convenient and rigorous approach should be based on the use of nondimensional quantities as described next. If U and L denote a characteristic velocity and a characteristic length, respectively, the nondimensionalization of u and x with respect to U and L , respectively, those of t , p and ρ with respect to L/c_e , $p_e = \rho_e c_e^2$ and ρ_e , respectively, then Eqs. (1) and (4) can be written as

$$\rho_t + M (\rho u)_x = 0, \quad (15)$$

$$M \rho (u_t + M u u_x) = -p_x + \alpha^2 M S(u), \quad (16)$$

where

$$S(u) = \rho_x (u_{xt} + M u u_{xx}) + \rho (M u_x u_{xx} + u_{xxt} + M u u_{xxx}), \quad (17)$$

$M = U/c_e$ is the Mach number and, for convenience the same symbols have been used for both dimensional and nondimensional variables. Note that, in Eq. (16), α^2 is dimensionless and no equation of state has been specified yet.

As stated in the Introduction, the validity of the Lagrangian-averaged Euler model employed in this paper hinges on the assumption that α^2 be small; if α^2 is $O(M^m)$, with $m \geq 1$, then Eq. (16) shows that the contribution of the α^2 -terms occur at higher-order in an asymptotic (power series) expansion of the velocity, pressure and density fields in terms of the Mach number, provided that this number is much smaller than unity. For $m = 0$, however, the effects of the α^2 -terms appear at the same order as $(\rho u)_x$ and ρu_t in the continuity and linear momentum equations, respectively, when power series expansions (in terms of the Mach number) are used for the velocity, pressure and density fields. Moreover, Eq. (17) implies that the leading-order terms of $S(u)$ in an asymptotic power series expansion are $\rho_x u_{xt}$ and ρu_{xxt} for low Mach numbers.

In what follows, we shall assume that $\alpha^2 = O(1)$, the Mach number is small, the second, third and fifth terms of Eq. (17) can be neglected (compared with the first and fourth terms) and revert to dimensional quantities. Moreover, by assuming that the density and pressure can be written as the sum of their equilibrium and fluctuating (acoustic) values, substitution of these values into Eqs. (1) and (6) but with the conditions imposed above yield the following wave equation

$$\rho'_{tt} - p'_{xx} = \alpha^2 \rho'_{xxt}. \quad (18)$$

For a barotropic fluid, $p' = \rho' c_e^2$, the substitution of which into Eq. (18) yields a Love's equation analogous to that of Eq. (13). For $\alpha^2 = O(M^m)$, with $m \geq 1$, the linear wave equation is obtained.

2.4 Asymptotic Lagrangian-averaged Euler models: Stokes' acoustic equation

If, in Eq. (2), the term $S(u)$ is replaced by $S(u) = \rho u_{xx}$, where α^2 has dimensions equal to length squared over time, and an analysis similar to the one described in the previous section is carried out for a barotropic fluid, the following equation results

$$\rho'_{tt} - c_e^2 \rho'_{xx} = \alpha^2 \rho'_{xxt}. \quad (19)$$

which is also referred to as Stokes' acoustic equation. This equation was derived by Stokes in 1845 [17] in his studies of acoustic wave propagation in a viscous fluid; if an analogy between Eq. (18) and that derived by Stokes is made, then $\alpha^2 = 4\mu/3\rho_e$ where μ denotes the shear dynamic viscosity of the fluid [18] which is assumed to be constant. This implies that, under the assumptions made at the beginning of this Section, a suitable chosen Lagrangian-averaged Euler model yields the same acoustic equation as that for one-dimensional, viscous flow. Both Stokes [17] and Rayleigh [19] obtained its corresponding dispersion relation which can be written as

$$\omega^2 + i\alpha^2 k^2 \omega - c_e^2 k^2 = 0, \quad (20)$$

which for real wavenumbers implies that the wave frequency is a complex quantity, the real and imaginary components of which, i.e., ω_R and ω_I , respectively, satisfy

$$2\omega_R \omega_I + \alpha^2 k^2 \omega_R = 0, \quad (21)$$

$$\omega_R^2 - \omega_I^2 - \alpha^2 k^2 \omega_I - c_e^2 k^2 = 0. \quad (22)$$

One of the solutions of Eq. (21) is $\omega_R = 0$, for which the fact that ω_I must be real implies through Eq. (22) that $\alpha^2 k^2 \geq 2c_e k$; this means that the wave number must be larger than or equal to $2c_e/\alpha^2$, the inverse of which is really the Stokes' length scale. The other solution of Eq. (21) is $\omega_I = -\alpha^2 k^2/2$, and the fact that ω_R must be real implies through Eq. (22) that $\alpha^2 k^2 \leq 2c_e k$, which is the opposite inequality to one discussed above.

Although some authors [20-22] have claimed that the solution to the Stokes' acoustic equation does not satisfy causality in the strict sense, i.e., a propagating pulse does not have a sharp front but extends asymptotically towards plus and minus infinity, some recent studies [23] have shown that all transient solutions of Stokes' equation are perfectly physical in their behavior and, in particular, no infinite wave speed is implied. This result is consistent with the dispersion relation (cf. Eq. (20)) which indicates that the phase speed diverges as the frequency increases but the attenuation or damping increases without bound as the frequency is increased and completely suppresses the infinitely fast, nonphysical Fourier components of the acoustic field.

It is worth noting that Stokes' equation also appears in conduction heat transfer when there are delays between the temperature gradient and the heat flux. For one-dimensional heat conduction, the first law of thermodynamics or the energy conservation equation in the absence of sources/sinks can be expressed as

$$\rho C T_t = -q_x, \quad (23)$$

where C and T denote the specific heat and the temperature, respectively, and q denotes the heat flux. When the heat flux is assumed to obey Fourier's law, then $q(t, x) = -k T_x(t, x)$, where k denotes the thermal conductivity.

Substitution of Fourier's law into Eq. (23) results in a one-dimensional parabolic equation which has an infinite speed of propagation; however, if there are time lags or delays between the heat flux and the thermal gradients, then Fourier's law may be modified and written as

$$q(t + \tau_q, x) = -k T_x(t + \tau_T, x), \quad (24)$$

where τ_q and τ_T denote the time lags for the heat flux and temperature, respectively. If the time lags are much smaller than the observation time, a simple Taylor's series expansion of Eq. (24) yields

$$q(t, x) + \tau_q q_x(t, x) = -k [T_x(t, x) + \tau_T T_{tx}(t, x)], \quad (25)$$

where higher-order terms have been neglected.

Partial differentiation of Eq. (25) with respect to x and use of Eq. (23) together with the assumption that ρ , C , k , and τ_q and τ_T are constants, yield

$$\rho C \tau_q T_{tt} - k T_{xx} = -\rho C T_t + k \tau_T T_{txx}, \quad (26)$$

which is a damped Stokes' equation and the damping comes through the thermal inertial of the material. As the time lags tend to zero, Eq. (26) tends to the conventional (parabolic) heat transfer equation. Moreover, the hyperbolic part, i.e., the left-hand side, of Eq. (26) is characterized by a finite wave speed the square of which is $k/\rho C \tau_q$ which depends on the thermal diffusivity, the characteristic damping time is τ_q and the analogue to α^2 in Eq. (19) is $(k/\rho C) (\tau_T/\tau_q)$ which is the product of the thermal diffusivity times the ratio of the time lag for the temperature to the time lag of the heat flux. Equation (26) is also valid in three dimensions provided that the term T_{xx} is replaced by the Laplacian of T .

3 Generalization

In this section, we consider an acoustic wave equation which includes nonlinearities, advection, damping and the Stokes' and Love's contributions derived in previous sections. This equation may be written as

$$u_{tt} + \beta u_t + \varphi(u) u_x = \mu u_{xx} + \delta u_{xxt} + \vartheta u_{xxxx}, \quad (27)$$

where β is the damping coefficient, $\varphi(u)$ is a nonlinear function of u , μ (assumed to be constant) is analogous to the square of the speed of sound, and δ and ϑ are constant terms that account for Stokes' and Love's contributions, respectively. In what follows, we shall be mainly concerned with $\varphi(u) = \alpha + \varepsilon u$, where the coefficients α and ε are constant and are associated with linear and nonlinear convection/advection. Equation (27) has an analytical solution provided that $\varepsilon = 0$. When this is not the case, Eq. (27) is nonlinear and must be solved numerically as indicated in the next section.

In order to solve numerically Eq. (27), we first introduce a new dependent variable $v \equiv u_t$, so that Eq. (27) may be written as

$$v_t + \beta v + \varphi(u) u_x = \mu u_{xx} + \delta v_{xx} + \vartheta v_{xxt}, \quad (28)$$

and then define $F \equiv u_x$, $G \equiv u_{xx}$, and $L \equiv v_{xx}$, so that Eq. (28) may be written as

$$v_t + \beta v + \varphi(u) F = \mu G + \delta L + \vartheta L_t. \quad (29)$$

Equation (29) and $v \equiv u_t$ have been discretized by means of a trapezoidal (Crank-Nicolson) rule in time which is second-order accurate and the nonlinear term $\varphi(u)$ was linearized with respect to time, so that the discretization of Eq. (29) results in the following linear algebraic equation

$$v^{n+1} - v^n + \beta \Delta t \frac{v^{n+1} + v^n}{2} + P = \mu \Delta t \frac{G^{n+1} + G^n}{2} + \delta \Delta t \frac{L^{n+1} + L^n}{2} + \vartheta (L^{n+1} - L^n), \quad (30)$$

where

$$P = \frac{\Delta t}{2} (2 \varphi^n F^n + \sigma^n F^n (u^{n+1} - u^n) + \varphi^n (F^{n+1} - F^n)), \quad (31)$$

and $P = d\varphi/du$.

The first- and second-order spatial derivatives have been discretized by means of three-point, fourth-order accurate, compact methods as

$$F_i^n = u_x(t^n, x_i) \approx \frac{1}{2 \Delta x} \frac{u_{i+1}^n - u_{i-1}^n}{1 + \delta^2/6} + O(\Delta x^4), \quad (32)$$

$$G_i^n = u_{xx}(t^n, x_i) \approx \frac{1}{\Delta x^2} \frac{\delta^2 u_i^n}{1 + \delta^2/12} + O(\Delta x^4), \quad (33)$$

which can, in turn, be written as

$$\frac{1}{6} (F_{i+1}^n + 4 F_i^n + F_{i-1}^n) = \frac{1}{2 \Delta x} (u_{i+1}^n - u_{i-1}^n) \quad (34)$$

$$\frac{1}{12} (G_{i+1}^n + 10 G_i^n + G_{i-1}^n) = \frac{1}{\Delta x^2} (u_{i+1}^n - 2 u_i^n + u_{i-1}^n). \quad (35)$$

where Δt and Δx are the time step and the spatial step size, respectively, the subscript i denotes $x_i = i \Delta x$, and the superscript n corresponds to $t^n = n \Delta t$, and equally-spaced grids in space and time have been used. An analogous expression to Eq. (35) holds for L .

Use of Eq. (30) at the i -th grid point together with Eqs. (34) and (35) and the analogous equation to Eq. (35) for L provide a block-tridiagonal system of linear algebraic equations for u , F , G and L which can be solved by the method of Thomas for block tridiagonal matrices [24]. Although not shown here, the implicit method presented in this section is unconditionally (linearly) stable and provides the values of u and v and the first- and second-order spatial derivatives of u and the second-order spatial derivative of v .

4 Presentation of results

Some sample results obtained with the numerical method reported in the previous section are presented here. These results are mainly concerned with the propagation of an initially compact pressure pulse, its splitting into two pulses travelling in opposite directions and their damping.

Figure 1 shows that an initially Gaussian pressure pulse of high amplitude splits into two waves that propagate in opposite directions in accord with the second-order hyperbolic character of Eq. (27); owing to the second term in the left-hand side of this equation, the amplitude of these two waves

decreases as time increases due to damping. The pressure pulses that result from the splitting of the initial one do not propagate symmetrically due to advection which imposes a flow asymmetry.

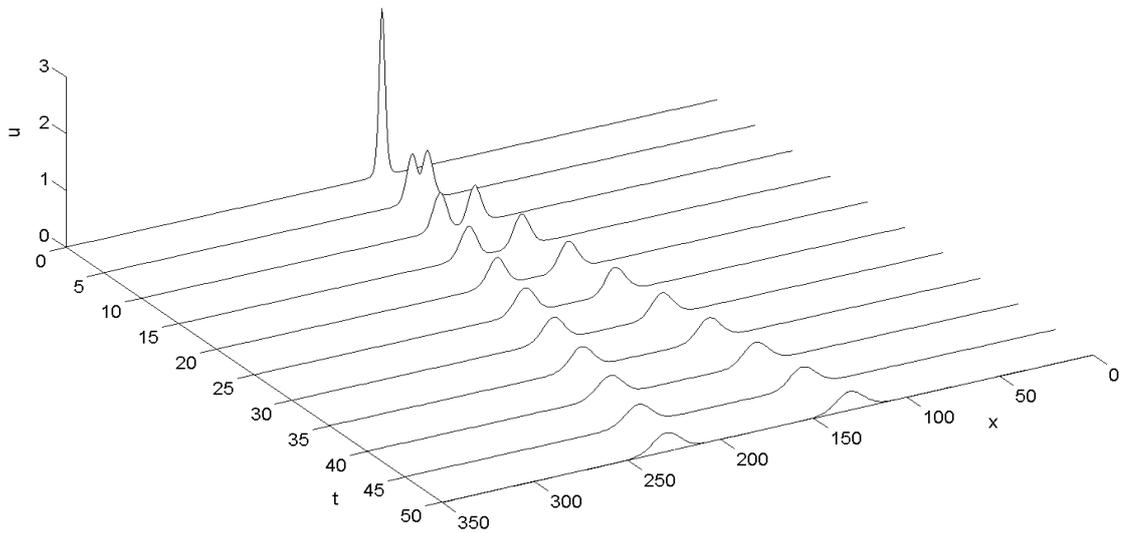


Figure 1 – Splitting and damping of a pressure pulse.

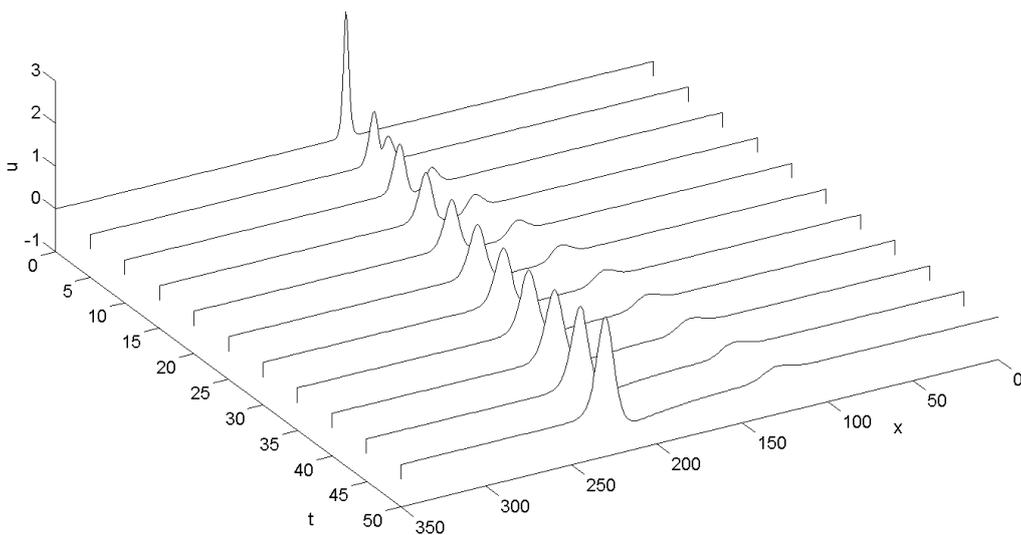


Figure 2 – Splitting, damping and growth of a pressure pulse.

Similar behavior to the one just described has been observed for other cases, at least initially as indicated in Figure 2. However, in this case, the asymmetry introduced by the advection terms causes that the pulse travelling towards the right increase its amplitude due to the nonlinearity of such terms, whereas that travelling to the left is damped. A noteworthy feature of Figure 2 is that u may become negative due to the pressure pulse splitting and the convective flow. Although as indicated above, the amplitude of the right-travelling wave pulse seems to increase as time increases, it has been found that, eventually, the steepening effects associated with the nonlinearities balance the ones associated with dispersion and a solitary wave is formed. Such a solitary wave may present an oscillatory tail due to the splitting suffered by the initial pressure pulse, and its amplitude decreases whereas its frequency increases as time increases.

If the damping is sufficiently large, it has been observed that the initial pressure pulse tries to split into two pulses, but, if the convection terms are sufficiently large, such splitting does not take place; instead, the pressure distribution exhibits a knee that is broadened due to damping, whereas the front of the wave steepens on account of the nonlinearities. However, if damping effects are stronger than those associated with steepening, the wave's front eventually decreases its slope/steepness and a broad pressure pulse results as indicated in Figure 3. This figure clearly shows the stages that a pressure pulse undergoes in the presence of convection and damping.

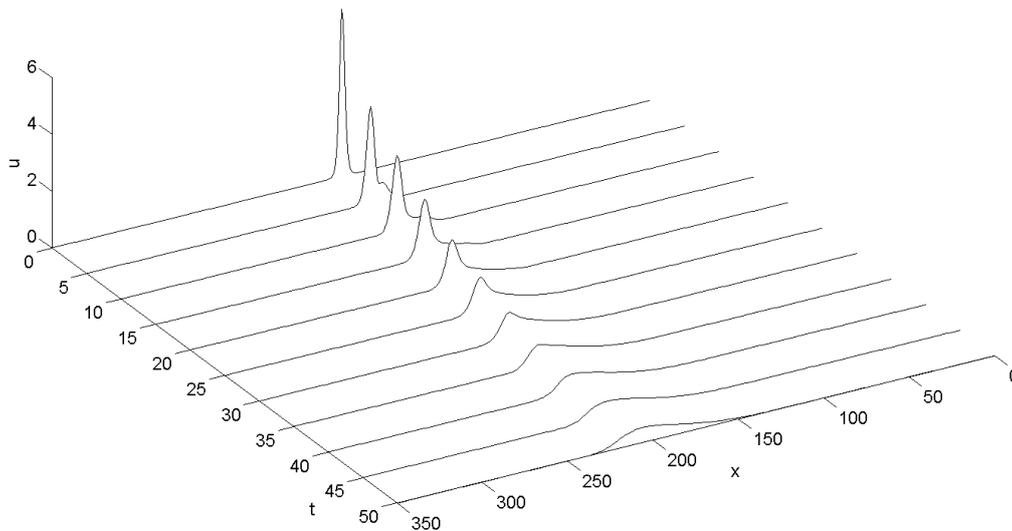


Figure 2 – Splitting, steepening and damping of a pressure pulse.

A comparison between the results presented in Figures 1-3 clearly shows that there is a competition between the nonlinear convection terms that try to increase the steepness of the pressure field and those associated with dispersion and damping or dissipation. Depending on which one wins such a competition, one may observe a complete damping of the initial pressure pulse, the splitting of the initial pressure pulse into two waves travelling towards the right and left, or a steepening of the wave's front with an oscillatory tail.

5 Conclusions

A Lagrangian-averaged one-dimensional Euler compressible flow formulation that includes a (small) constant coarseness parameter and reduces to the one-dimensional Euler equation when the coarseness parameter is set to zero has been used to derive its corresponding acoustic equations. It has been shown that the coarseness term of this model can be written in a simple conservation-law form and that, when this coarseness term is replaced by that corresponding to the Euler equations, the resulting acoustic equation contains a fourth-order spatial derivative.

It has also been shown that, if the smallness of the coarseness parameter in asymptotic power series expansions of the pressure, density and velocity field, the resulting leading-order acoustic equation is that of Love, contains a fourth-order derivative (twice with respect to space and twice with respect to time) and models elastic phenomena as well the propagation of sound in bubbly liquids.

It is also shown that a simple modification of the coarseness term of the Lagrangian-averaged Euler model results in Stokes' equation which contains a third-order derivative (twice with respect to space and once with respect to time) and has been previously derived from the one-dimensional Navier-Stokes equations for one-dimensional flows under the assumption that the fluid motion is isothermal.

A generalized acoustic equation which includes damping, nonlinear advection, and the Love's and Stokes's contributions described in this paper has also been proposed and studied numerically by means of a second-order accurate time-linearized Crank-Nicolson method and three-point, fourth-order accurate compact discretizations in space. The results of this method indicate that an initial Gaussian pressure distribution may split into two travelling pulses the amplitudes of which decrease as time increases; the left travelling pulse may be damped while the amplitude of the right travelling one may increase until it becomes a solitary wave of constant amplitude; or, the front of the right travelling pulse may steepen but the pulse amplitude decreases whereas its width increases as time increases.

Acknowledgements

The research reported in this paper was supported by Project FIS2009-12894 from the Ministerio de Ciencia e Innovación of Spain and fondos FEDER.

References

- [1] Bhat, H.; Fetecau, R. C.; Marsden, J. E.; Mohseni, K.; West, M. Lagrangian averaging for compressible fluids. *SIAM J. Multiscale Modeling and Simulation*, Vol. 3, 2005, pp. 818-837.
- [2] Holm, D. D. Fluctuation effects in 3D Lagrangian mean and Eulerian mean fluid motion. *Physica D*, Vol. 133, 1999, pp. 215-269.
- [3] Holm, D. D.; Marsden, J. E.; Ratiu, T. S. Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Physical Review Letters*, Vol. 349, 1998, pp. 4173-4177.
- [4] Mohseni, K.; Kosović, B.; Shkoller, S.; Marsden, J. E. Numerical simulations of the Lagrangian averaged Navier-Stokes equations for isotropic turbulence. *Physics of Fluids*, Vol. 15, 2003, pp. 524-544.
- [5] Marsden, J. E.; Shkoller, S. Global well-posedness for the LANS- α equations on bounded domains. *Proceedings of the Royal Society of London, Series A*, Vol. 359, 2001, pp. 1449-1468.
- [6] Bhat, H.; Fetecau, R. C. Lagrangian averaging for the 1D compressible Euler equations. *Discrete and Continuous Dynamical Systems-Series B*, Vol. 6 (5), 2006, pp. 979-1000.

- [7] Keiffer, R. S.; McNorton, R.; Jordan, P. M.; Christov, I. M. Dissipative acoustic solitons under a weakly-nonlinear, Lagrangian-averaged Euler- α model of single-phase lossless fluid. *Wave Motion*, Vol. 48 (8), 2011, pp. 782-790
- [8] Love, A. E. H. *Mathematical Theory of Elasticity*, Dover Publications, Inc., New York, 1944.
- [9] Lesser, M. B.; Seebass, R. The structure of a weak shock wave undergoing reflexion from a wall. *Journal of Fluid Mechanics*, Vol. 31, 1968, pp. 501-528.
- [10] Crighton, D. G. Model equations of nonlinear acoustics. *Annual Review of Fluid Mechanics*, Vol. 11, 1979, pp. 11-33.
- [11] Naugolnykh, K.; Ostrovsky, L. *Nonlinear Wave Processes in Acoustics*, Cambridge University Press, New York, 1998.
- [12] Makarov, S.; Ochmann, M. Nonlinear and thermoviscous phenomena in acoustics, Part I. *Acustica-Acta Acustica*, Vol. 82, 1996, 579-606.
- [13] Makarov, S.; Ochmann, M. Nonlinear and thermoviscous phenomena in acoustics, Part II. *Acustica-Acta Acustica*, Vol. 83, 1997, 197-222.
- [14] Van Wijngaarden, L. On the equations of motion for mixtures of fluid and gas bubbles. *Journal of Fluid Mechanics*, Vol. 33, 1968, pp. 465-474.
- [15] Jordan, P. M.; Feuillade, C. On the propagation of transient acoustic waves in isothermal bubbly liquids. *Physics Letters A*, Vol. 350, 2006, pp. 56-62.
- [16] Barenblatt, G. I.; Zheltov, Iu. P.; Kochina, I. N. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks (strata). *Prikladnaya Matematika i Mekhanika (PMM), Journal of Applied Mathematics and Mechanics*, Vol. 24 (5), 1960, pp. 852-864.
- [17] Stokes, G. G. On the theories of internal friction on fluids in motion. *Transactions of the Cambridge Philosophical Society*, Vol. 8, 1845, pp. 287-305.
- [18] Kinsler, L. E.; Frey, A. R.; Coppens, A. B., Sanders, J. V. *Fundamentals of Acoustics*, 4th edition, Wiley, New York, 1999.
- [19] Strutt, J. W. (Lord Rayleigh). *The Theory of Sound*, Dover Publications, Inc., New York, 1945.
- [20] Blackstock, D. T. Transient solution for sound radiated into a viscous fluid. *Journal of the Acoustical Society of America*, Vol. 41 (5), 1967, pp. 1312-1319.
- [21] Norwood, F. R. Propagation of transient sound signals into a viscous fluid. *Journal of the Acoustical Society of America*, Vol. 44 (2), 1968, pp. 450-457.
- [22] Jordan, P. M.; Meyer, M. R.; Puri, A. Causal implications of viscous damping in compressible fluid flows. *Physical Review E*, Vol. 62, 2000, pp. 7918-7926.
- [23] Buckingham, M. J. Causality, Stokes' wave equation, and acoustic pulse propagation in a viscous fluid. *Physical Review E*, Vol. 72, 2005, 026610-1/9.
- [24] Morton, K. W.; Mayers, D. F. *Numerical Solutions of Partial Differential Equations*, Cambridge University Press, New York, 1994.