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PHYSICAL ORIGINS OF WILLIS COUPLING IN ACOUSTIC METAMATERIALS

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Abstract
It is convenient to study wave phenomena in inhomogeneous media using dynamic effective material properties. In addition to the traditional properties (effective wavenumber, impedance, density, and compressibility), recent research has shown that inhomogeneous media with asymmetric unit cells and/or composed of a periodic lattice require effective properties in the long wavelength limit that couple the averaged volume-strain and momentum fields. This behavior, known as Willis coupling in elasticity, is analogous to bianisotropy in electromagnetism. The present work demonstrates the microscale and mesoscale origins of macroscopic coupling, and that the resulting effective properties satisfy passivity and causality.

Keywords: acoustic bianisotropy, acoustic metamaterials, Willis-material.

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1 Introduction

Recent interest in metamaterials has provided a new impetus for the study of homogenization schemes that account for multiscale dynamics. One very important result of general homogenization in elastodynamics that has received relatively little attention to date, is the fact that the effective dynamic properties of inhomogeneous elastic materials results in constitutive relations that couple strain to momentum and stress to velocity in the frequency domain. This type of coupling has come to be known as Willis coupling since it was first formally introduced by J.R. Willis in the 1980's [1]. Recent work by Milton and Willis [2] outline the development of this theory and present it as analogous to bianisotropy in electromagnetism. In general, bianisotropy refers to the coupling of electrical responses to magnetic excitation and magnetic responses to electrical excitation, as in Equation (1), and provides key insights into the origins of Willis coupling.

The study of bianisotropy in electromagnetism began over 200 years ago with the observation of "optical activity" in some crystals, in which the polarization of linear polarized light rotates as it

propagates through the media [3]. This material response is reciprocal and is also generally known as chirality. In the mid-20th century, it was shown that certain magnetic materials exhibit magnetization proportional to electrical excitation. This became known as the magnetoelectric effect and is nonreciprocal [3]. Both of these effects were demonstrated in engineered materials in the first half of the 20th century, and the general response of isotropic materials displaying both chirality and magnetoelectric effect (*i.e.* bi-isotropic materials) can be described using the following coupled linear electromagnetic constitutive relations

$$D = \varepsilon E + \left(\kappa^{\mathrm{nr}} + \kappa^{\mathrm{or}} + i\kappa^{\mathrm{er}}\right) H \quad \text{and} \quad B = \mu H + \left(\kappa^{\mathrm{nr}} + \kappa^{\mathrm{or}} - i\kappa^{\mathrm{er}}\right) E. \tag{1}$$

In these expressions, κ^{er} describes the chirality of the medium, and κ^{nr} describes the non-reciprocal magnetoelectric effect, also known as the Tellegen parameter. More generally, κ^{nr} can describe several nonreciprocal coupling mechanisms including moving media and time-varying media. κ^{er} contains coupling mechanisms which are even in wavenumber and reciprocal, hereafter referred to as even coupling, including substrate effects [4] and artificial magnetism from structures such as split-ring resonators and omega particles. The parameter κ^{or} is odd in wavenumber and reciprocal, hereafter referred to as odd coupling, and describes non-local lattice effects [5]. A major result of [5] is that neglecting κ^{or} in the homogenization of periodic arrays can result in effective properties that do not satisfy the restrictions placed on material response to external fields by passivity and causality.

The analog to chirality in elastodynamics was first proposed and demonstrated by Varadan et al. [6], who restricted their study to transverse (shear) wave fields. However, a surprising result of Willis' work is that it predicts coupling for both transverse *and* longitudinal waves [1]. Given that fluids only support longitudinal waves, one must conclude that coupling exists in acoustics. Reciprocal coupling in 1D longitudinal waves has recently been calculated by several groups for periodic media [7,8], and it has been noted that even coupling only occurs when the unit cell lacks "reflective symmetry" [7]. It was demonstrated in [9] that the "real part" of the coupling computed in [7,8] was due to multiple scattering in the lattice, *i.e.* odd coupling. The multiple scattering analysis in [9] employed a homogenization scheme used by Alù [10] in an analogous study of electromagnetic metamaterials. It is worth mentioning that analogous nonreciprocal coupling has also been recently demonstrated for elastic waves using a layered media with time-varying properties [11], and initial measurements and applications of even coupling have been proposed [12].

The primary focus of this work is to provide physical insight into the origin of two observed reciprocal coupling mechanisms for longitudinal waves in acoustic metamaterials. While the homogenization procedures in references [7,8] and others can provide exact effective material parameters, the physical mechanisms are difficult to discern. This work is therefore limited to the demonstration of reciprocal coupling in 1D periodic acoustic metamaterials. This is achieved via source-driven homogenization and a self-consistent multiple scattering scheme as in references [9,10]. The resulting effective properties contain field coupling and satisfy passivity and causality.

2 Source-driven homogenization

This section summarizes the source-driven homogenization procedure of [9,10], which yields closed form expressions for effective material properties that are unique for any (ω, k) pair.

2.1 Field definitions

Consider a lossless homogeneous fluid characterized by mass density, ρ_0, and compressibility, β_0. This fluid contains an externally controlled source distribution modeled with complex time-harmonic body forces (dipole sources), f_{ext}, and volume (monopole) sources, q_{ext}, imposing an $\exp(ikx - i\omega t)$ dependence. Although fictitious, the continuous source distribution guarantees uniqueness of fields and permits the ability to impose any desired (ω, k) pair [10]. The complex amplitudes of the acoustic pressure and particle velocity fields are determined from the momentum and mass conservation equations;

$$ikp_{\text{ext}} = i\omega\rho_0 u_{\text{ext}} + f_{\text{ext}} \quad \text{and} \quad iku_{\text{ext}} = i\omega\beta_0 p_{\text{ext}} + q_{\text{ext}}. \tag{2}$$

Introducing an array of inhomogeneities into the fluid and averaging over the unit cell (as described in [10]), the conservation relations for the averaged fields can be expressed in terms of the background material properties, ρ_0 and β_0, averaged dipole and monopole polarizations, D_{av} and M_{av}, respectively, as

$$ikp_{\text{av}} = i\omega\rho_0 u_{\text{av}} + i\omega D_{\text{av}} + f_{\text{ext}} \quad \text{and} \quad iku_{\text{av}} = i\omega\beta_0 p_{\text{av}} - i\omega M_{\text{av}} + q_{\text{ext}}. \tag{3}$$

As a result of the source distribution, the same (ω, k) pair is maintained in the effective medium. From Equations (3), constitutive relations can be defined that relate the averaged momentum density to the averaged particle velocity and dipole polarization and averaged volume strain to the averaged pressure and monopole polarization,

$$\mu_{\text{av}} = \rho_0 u_{\text{av}} + D_{\text{av}} \quad \text{and} \quad \varepsilon_{\text{av}} = -\beta_0 p_{\text{av}} + M_{\text{av}}, \tag{4}$$

respectively. From these constitutive relations, it is clear that the averaged polarizations determine how the effective properties of the metamaterial differ from the background media. The derivation of the averaged polarizations for a 1D periodic metamaterial is outlined in the next few sections.

2.2 Inhomogeneity response

For low volume fractions of inhomogeneities, the inhomogeneity present in the *n*th unit cell can be modeled by equivalent point dipole, d_n, and monopole, m_n, moments in terms of local fields and polarizabilities, α, as

$$\frac{d_n}{\rho_0} = \alpha_{\text{d}} u_{\text{loc},n} - i\alpha_{\text{c}} \frac{1}{Z_0} p_{\text{loc},n} \quad \text{and} \quad \frac{m_n}{\beta_0} = -\alpha_{\text{m}} p_{\text{loc},n} - i\alpha_{\text{c}} Z_0 u_{\text{loc},n}. \tag{5}$$

Polarizabilities have units of volume and can be determined from the scattering matrix of a single inhomogeneity, and local fields correspond to the fields present at the center of the inhomogeneity in its absence. In addition to the dipole, α_{d}, and monopole polarizabilities, α_{m}, Equations (5) contain a coupling polarizability, α_{c}, which maps the scalar pressure field to the dipole moment and the vector velocity field to the monopole moment. It is here at the microscale that even coupling originates from asymmetry.

To provide physical insight into this phenomenon, an example asymmetric inhomogeneity is presented in Figure 1 which consists of three layers (referred to by 1, 2, and 3 from left-to-right) along with its equivalent mechanical system. Layers 1 and 3 are assumed rigid with mass per unit area, $\rho_1 l_1$ and $\rho_3 l_3$, respectively, whereas layer 2 has negligible mass and a stiffness per unit area, $1/\beta_2 l_2$. Damping due to sound radiation are modeled as dashpots. As demonstrated by the equivalent mechanical

system, a pure monopole excitation would be represented by equal time-harmonic forces applied to the two masses in opposite directions. For $\rho_1 l_1 \neq \rho_3 l_3$, the velocity drops across the two dashpots will not be exactly equal and in opposite directions, implying that the field scattered by this simple asymmetric inhomogeneity contains contributions from both monopolar and dipolar motion. Similarly, a dipole excitation will also excite dipole and monopole moments. This is the origin of even coupling discussed in the introduction.

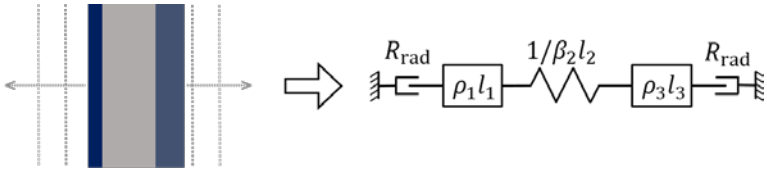

Figure 1 – Microscopic origin of even coupling.

2.3 Lattice interaction

To quantify the interaction between unit cells, the local fields are expressed as the sum of externally imposed fields and scattered contributions from all other inhomogeneities using the equivalent point dipole and monopole moments. For simplicity, choosing $n = 0$ as the reference, the local fields can be expressed in terms of interaction coefficients, *C*, as

$$u_{\text{loc}} = u_{\text{ext}} + C_{\text{d}} \frac{d_0}{\rho_0} - C_{\text{c}} \frac{1}{Z_0} \frac{m_0}{\beta_0} \quad \text{and} \quad p_{\text{loc}} = p_{\text{ext}} - C_{\text{m}} \frac{m_0}{\beta_0} + C_{\text{c}} Z_0 \frac{d_0}{\rho_0}. \tag{6}$$

The interaction coefficients can be calculated as geometric series and are given in [9]. Equations (6) demonstrate presence of coupling between dipolar and monopolar motion at the lattice level, or mesoscale. From Equations (5) and (6), dipole and monopole moments will be excited for every inhomogeneity because local velocity and pressure will be non-zero at each inhomogeneity from multiple scattering, regardless of externally imposed fields. This mesoscale coupling is independent of the coupling observed at the microscale in the previous section, and since it is related to the gradient of the Green's function, as shown in [9], it will be odd in wavenumber.

2.4 Effective properties

To determine the effective properties, the averaged polarizations, Equation (4), must be related to the microscale responses and mesoscale interactions. From [10], The averaged dipole and monopole polarizations are directly related to dipole and monopole moments, respectively, of a unit cell as $D_{\text{av}} = d_0 / V$ and $M_{\text{av}} = m_0 / V$, where $V = SL$ is the unit cell volume. Combining Equations (2), (3), (5), and (6), and solving for D_{av} and M_{av}, it becomes clear that averaged dipole and monopole polarizabilities are both dependent on the averaged particle velocity and averaged pressure. This leads to the constitutive relations in Equation (4), to take on the same form as the bi-isotropic relations, (1);

$$\mu_{\text{av}} = \rho_{\text{eff}} u_{\text{av}} - (\chi^{\text{o}}_{\text{eff}} + i\chi^{\text{e}}_{\text{eff}}) p_{\text{av}} \quad \text{and} \quad \varepsilon_{\text{av}} = \beta_{\text{eff}} p_{\text{av}} + (\chi^{\text{o}}_{\text{eff}} - i\chi^{\text{e}}_{\text{eff}}) u_{\text{av}}. \tag{7}$$

The effective mass density is most closely related to dipole-dipole interactions and effective compressibility to monopole-monopole interactions at the microscale and mesoscale. The odd, $\chi^{\text{o}}_{\text{eff}}$, and even, $\chi^{\text{e}}_{\text{eff}}$, coupling parameters, however, have more complicated relations. $\chi^{\text{o}}_{\text{eff}}$ is proportional to C_{c} at the mesoscale and monopole and dipole contributions, and $\chi^{\text{e}}_{\text{eff}}$ is proportional α_{c} at the microscale and monopole and dipole contributions.

3 Conclusions

The microscale and mesoscale origins of Willis coupling have been demonstrated for 1D periodic acoustic metamaterials using source-driven homogenization and a self-consistent multiple scattering scheme. The approach is valid for resonant and non-resonant layered inhomogeneities in a background fluid.

Acknowledgements

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