NONLINEAR STRING EFFECTS ON THE THREE-DIMENSIONAL STRING/SOUNDBOARD COUPLED DYNAMICS OF A TWELVE-STRING PORTUGUESE GUITAR

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ABSTRACT
The Portuguese guitar is a pear-shaped instrument with twelve metal strings (six courses), descendant from the renaissance European cittern. This instrument is closely associated with the Portuguese traditional music called “Fado” but it has recently started to play a considerable role among urban Portuguese musicians. Unlike common guitars, this guitar has a bent soundboard (arched top) with a bridge somewhat similar, although smaller in size, to the bridge of a violin. Based on our recently developed physical model of the Portuguese guitar, the present paper introduces a nonlinear formulation for the string dynamics in order to reproduce some musically significant effects obtained with real-life instruments. Nonlinear string vibrations are described by the Kirchoff-Carrier equations while the instrument body is idealized by a FEM model of the top plate of a typical Portuguese guitar.

The coupling of the strings to the structural modes of the instrument body is ensured by the bridge action modeled through a simple geometrical rationale. Emphasis here is on illustrating the range of phenomena induced by the nonlinear behaviour of the strings and asserting the correct behaviour of our computational modal-based model.

1 Introduction

Although recent models for string instruments embody many aspects of complex nature such as the interaction between the excitation, the strings, the soundboard and the surrounding air, one still debatable issue is the origin of the polarization change commonly observed on musical strings under transient excitations. If the non-planar dynamics naturally raises fundamental questions regarding the behaviour of a stretched string, it has also significant implications for the sound produced by string instruments because both polarizations radiate sound. For musical instruments such as the piano and some type of guitars including the Portuguese guitar, an obvious candidate for the source of interaction is the bridge action which affects the motions of the strings attached to it by coupling the body vibrations through some rotation. In a more general context, many theoretical and experimental investigations of the free vibrating string have attested that mode-coupling effect arises from nonlinear geometrical effects [1–3]. However, other physical factors can also be pointed as discussed in a paper by Elliott [4], namely the nature of the end conditions and the nonuniformity in string material, but their effects are rather difficult to quantify reliably. Apart from its influence on the string polarization, the nonlinearity is also of interest from the musical point of view as it leads to audible significant effects.

This certainly explains the recent increasing interest in modelling large-amplitude string vibrations for sound synthesis purpose [5–8]. In particular, the intrinsic nonlinearity leads to the spectral changes observed for different amplitudes of excitation, to the characteristic pitch glide effect as well as to some beats phenomena in the sound produced [9].

In pursuit of the goal of modeling the Portuguese guitar by physical-based methods, we now relax the linearity assumption for the string behaviour used in [10] and introduce, in this paper, a nonlinear formulation for the strings dynamics. Physically, it consists in accounting for the local changes in string tension which inevitably occur for large amplitude vibrations. This excites the longitudinal motion of the

string and in turn couples the two perpendicular transverse string motions. By including this mode-coupling, we therefore expect the model to be capable of achieving more detailed simulations and producing more realisitic synthetic sounds. The other latent objective of this work is to discuss various source mechanisms for the string mode-coupling effect, in particular by comparing the influence of the geometrical nonlinearity of the string with the bridge action. Indeed, in [10], we have presented numerical simulations which include twelve strings, an idealized guitar body and a bridge simply modelled through geometrical rationale. The results showed that the bridge action does affect the strings motions but the main conlusion is that its influence remains marginal.
On the technical ground, there is a wide variety of possible approaches to model nonlinear string vibrations according to the level of refinements retained for the transverse/longitudinal mode coupling - see [11] for a detailed discussion. The Kirchoff-Carrier string model which is used in this work is particularly attractive for its simplicity when compared to other approximate models stemming from the "geometrically exact" model for vibrating strings, see [5, 8, 11, 12]. This model does not describe precisely the dynamics of the longidutinal motion. Instead, it regards its effects on the transverse string motions as coming from quasi-static tension variations originated by the net increase in string length due to the transverse motions. As discussed in Sec. 3.2, this clearly limits the validity of the model but from the numerical point of view, it is more convenient than other techniques stemming from the exact formulation which require lengthy computations with thousands of coupling terms to calculate at each time step. As a consequence, this model has been used in many studies dealing with sound synthesis of string instruments, but to our best knowledge, none of them concerns the full modeling of a real instrument as done in this work.
In this paper, we first remind the essential ingredients of our linear modeling of the Portuguese guitar and then present the governing equations for large-amplitude string motions. Some aspects of the numerical implementation are discussed and, finally, preliminary results are presented to highlight the range of phenomena.

2 Presentation of the developed Portuguese guitar model

Figure 1: António Chainho playing the Lisbon Portuguese Guitar.

The Portuguese guitar is a plucked string instrument with twelve metal strings, descendant from the renaissance European cittern. A detailed structure and acoustical analysis of the Portuguese guitar can be found in [13] but briefly, it has an arched soundboard, with a round soundhole and a specific violin-like *floating* bridge. The physical model presented here has been developed in [10] and it describes the main interaction involved in music performance. It includes the coupled dynamics of twelve strings supported by a bridge and an idealized instrument body described through the modal parameters of the top-plate of a real instrument stemming from FEM modal computation, with further simple models for the string/fret interaction and the pluck excitation. The model is entirely based on linear assumptions, for both the displacements and the interactions, and the bridge action is simply described by assuming a quasi-static response of its dynamics.

String dynamics The string model includes both polarisation, normal and parallel to the soundboard, and here concerns relatively small-amplitude vibrations only. We consider a set of $s = 1, \ldots, S$ perfectly flexible strings of total length L (from the *atadilho*, a small tailpiece at the end of the body of the instrument, to the neck) and density ρ_s, stretched to an axial tension T_0^s. The strings are rigidily fixed at both ends and stretched over the bridge, so that their natural frequencies are lower than the sounding frequencies defined by the active length of the string, i.e the distance between the bridge and the neck. The linear behaviour of the vibrating string is classically governed by the 1D-wave equation, for which a modal discretization leads to the set of ODEs:

$$m_n^{ys}\,\ddot{q}_n^{ys}(t) + c_n^{ys}\,\dot{q}_n^{ys}(t) + k_n^{ys} q_n^{ys}(t) = \mathcal{F}_n^{ys}(t) \tag{1a}$$
$$m_n^{zs}\,\ddot{q}_n^{zs}(t) + c_n^{zs}\,\dot{q}_n^{zs}(t) + k_n^{zs} q_n^{zs}(t) = \mathcal{F}_n^{zs}(t) \tag{1b}$$

Here, the time-dependent terms $q_n^{ys}(t)$ and $q_n^{zs}(t)$ are the modal amplitudes of the transverse string motions in the directions parallel and normal to the soundboard, so that the physical string motions in the $Y_s(x,t)$ and $Z_s(x,t)$ directions can be computed by modal superposition as:

$$Y_s(x,t) = \sum_{n=1}^{N_s} q_n^{ys}(t)\phi_n^{ys}(x)\,, \quad Z_s(x,t) = \sum_{n=1}^{N_s} q_n^{zs}(t)\phi_n^{zs}(x) \tag{2}$$

where $\phi_n^{ys}(x) = \phi_n^{zs}(x) = \sin(n\pi x/L)$ are the transverse mode shapes of the string in the two polarization planes, N_s being the size of each string modal basis. The other modal parameters are the modal mass, modal stiffness, modal damping value and circular modal frequency, given respectively by:

$$m_n^{\mathbf{X}s} = \rho_s L/2\,, \quad k_n^{\mathbf{X}s} = m_n^{\mathbf{X}s}(\omega_n^{\mathbf{X}s})^2\,, \quad c_n^{\mathbf{X}s} = 2 m_n^{\mathbf{X}s}\omega_n^{\mathbf{X}s}\zeta_n^{\mathbf{X}s} \quad \omega_n^{\mathbf{X}s} = \sqrt{\frac{T_0^s}{\rho_s S}}\Big(\frac{n\pi}{L}\Big) \tag{3}$$

where $\mathbf{X}s$ stands either for the Y and Z polarizations of string s. The forcing terms in Eqs. (1) stem from the projection of the external force $f(x,t)$ on each mode shape and is computed generically as

$$\mathcal{F}_n^{\mathbf{X}s}(t) = \int_0^L f(x,t)\phi_n^{\mathbf{X}s}(x)dx\,, \tag{4}$$

As described in the following, the external force field includes the local effects of (*a*) the finger/string interaction during the pluck, (*b*) the stopping fret when the musician presses the string on the fingerboard, and (*c*) the body vibration coupled at the bridge.

String/finger excitation The string/finger interaction is modeled very simply using a spring/dashpot model. The idea is to attach the finger to the string until the instant of release. The plucking action of the player during the sticking phase is therefore expressed in terms of two forces as:

$$F_Y^e(t) = -K_e\Big[Y(x_s^e,t) - Y^e(t)\Big] - C_e\Big[\dot{Y}(x_s^e,t) - \dot{Y}^e(t)\Big] \tag{5a}$$
$$F_Z^e(t) = -K_e\Big[Z(x_s^e,t) - Z^e(t)\Big] - C_e\Big[\dot{Z}(x_s^e,t) - \dot{Z}^e(t)\Big] \tag{5b}$$

where $F_Y^e(t)$ and $F_Z^e(t)$ are the forces interactions in the planes parallel and normal with respect to the soundboard, $Y(x_s^e,t)$ and $Z(x_s^e,t)$ are the string displacements in the two directions at the excitation location x_s^e, $\dot{Y}(x_s^e,t)$ and $\dot{Z}(x_s^e,t)$ being the corresponding velocities, $Y^e(t)$ and $Z^e(t)$ are the finger displacements in the two directions at the excitation location, with K_e and C_e the stiffness and damping coupling coefficients between the finger and the strings.

String/fret interaction In order to control the playing frequency, the musician presses the strings against the fingerboard with its left-hand fingers, which eventually prevents any string motion at the fret location, and thus shortens the active length of the vibrating string. As for the string/finger coupling, the interaction force exerted by the fret on the string at the fret location x_s^f is modelled by a penalty formulation, using two suitable coupling constants K_f and C_f and imposing a near-zero string displacement at x_s^f, according to:

$$F_Y^f(t) = -K_f Y(x_s^f, t) - C_f \dot{Y}(x_s^f, t) \tag{6a}$$

$$F_Z^f(t) = -K_f Z(x_s^f, t) - C_f \dot{Z}(x_s^f, t) \tag{6b}$$

where $F_Y^f(t)$ and $F_Z^f(t)$ are the force interactions in the two polarization planes $Y(x_s^f, t)$ and $Z(x_s^f, t)$ are the string displacements at the fret location, and $\dot{Y}(x_s^f, t)$ and $\dot{Z}(x_s^f, t)$ are the corresponding velocities.

String/bridge interaction The coupling between the string and the body of the violin first passes through the string/bridge interaction. Following Inácio *et al.* [14], as in Eqs. (5) and (6), we introduced a penalty model for this interaction by connecting the string to the bridge through a very stiff spring (with a damper to minimize residual local oscillations). The forces exerted by the body on a given string *s* are given by:

$$F_Y^b(t) = -K_b\Big[Y(x_s^b, t) - Y_s^b(t)\Big] - C_b\Big[\dot{Y}(x_s^b, t) - \dot{Y}_s^b(t)\Big] \tag{7a}$$

$$F_Z^b(t) = -K_b\Big[Z(x_s^b, t) - Z_s^b(t)\Big] - C_b\Big[\dot{Z}(x_s^b, t) - \dot{Z}_s^b(t)\Big] \tag{7b}$$

where K_b is the (high) stiffness coupling coefficient between the bridge and the strings, C_b is the damping coupling coefficient, $Y_s(x_s^b, t)$, $Z_s(x_s^b, t)$ and $\dot{Y}_s(x_s^b, t)$ and $\dot{Z}_s(x_s^b, t)$ are the string displacements and velocities at the bridge location x_s^b in both directions respectively. As presented in [10] in detail, the bridge motion at the anchoring location of any given string s, $Y_s^b(t)$ and $Z_s^b(t)$, can be computed from the bridge feet motion through the relation:

$$\begin{Bmatrix} Y_s^b(t) \\ Z_s^b(t) \end{Bmatrix} = \begin{pmatrix} \dfrac{z_s}{L_b} & -\dfrac{z_s}{L_b} \\ \dfrac{1}{2} - \dfrac{y_s}{L_b} & \dfrac{1}{2} + \dfrac{y_s}{L_b} \end{pmatrix} \begin{Bmatrix} Z_{b_1}(t) \\ Z_{b_2}(t) \end{Bmatrix} \tag{8}$$

and similarly for the velocities. Here, (y_s, z_s) refers to the string contact point coordinates in the plane of motion of the bridge, $L_b = y_{b_2} - y_{b_1}$ is the distance between the bridge feet and $Z_{b_1}(t)$ and $Z_{b_2}(t)$ are the displacements of the bridge feet, which are computed as:

$$Z_{b_1}(t) = \sum_{n=1}^{N_{SB}} q_n^{SB}(t)\phi_n^{SB}(x_{B_1}, y_{B_1}), \quad Z_{b_2}(t) = \sum_{n=1}^{N_{SB}} q_n^{SB}(t)\phi_n^{SB}(x_{B_2}, y_{B_2}) \tag{9}$$

where $\phi_n^{SB}(x_{B_1}, y_{B_1})$ and $\phi_n^{SB}(x_{B_2}, y_{B_2})$ are the mode shapes of the soundboard at the bridge feet locations (x_{B_1}, y_{B_1}) and (x_{B_2}, y_{B_2}).

Soundboard dynamics In a modal framework, the body response is described by the set of modal equations:

$$m_n^{SB}\,\ddot{q}_n^{SB}(t) + c_n^{SB}\,\dot{q}_n^{SB}(t) + k_n^{SB}\,q_n^{SB}(t) = \mathcal{F}_n^{SB}(t) \tag{10}$$

where m_n^{SB}, c_n^{SB} and k_n^{SB} are the soundboard modal parameters, and $q_n^{SB}(t)$ the soundboard modal responses ($n = 1, \ldots, N_{SB}$). As for the strings, the modal forces are obtained by projecting the bridge/soundboard interaction forces on the instrument body modal basis, yielding to:

$$\mathcal{F}_n^{SB}(t) = -F_{b_1}(t)\phi_n^{SB}(x_{b_1}, y_{b_1}) - F_{b_2}(t)\phi_n^{SB}(x_{b_2}, y_{b_2}) \tag{11}$$

where $F_{b_1}(t)$ and $F_{b_2}(t)$ are the vertical forces between the bridge feet and the soundboard. By assuming a quasi-static response of the bridge, these vertical forces are given by [10]:

$$F_{b_1}(t) = \frac{1}{2}\sum_{s=1}^{S} F^b_{Z_s}(t) + \frac{1}{L_b}\left[\sum_{s=1}^{S} F^b_{Y_s}(t)\, z_s - \sum_{s=1}^{S} F^b_{Z_s}(t) y_s\right] \tag{12a}$$

$$F_{b_2}(t) = \frac{1}{2}\sum_{s=1}^{S} F^b_{Z_s}(t) - \frac{1}{L_b}\left[\sum_{s=1}^{S} F^b_{Y_s}(t)\, z_s - \sum_{s=1}^{S} F^b_{Z_s}(t) y_s\right] \tag{12b}$$

For illustration, Figure 2 presents two spectrogams of the soundboard velocity obtained considering first a single string, and then the fully 12-string/bridge/body coupled model, for a vertical pluck of the lowest open string. Note the appearance of audible beats in the sound which shows the sympathetic excitation of slightly mistuned harmonics of the different string subsystems due to their coupling at the bridge.

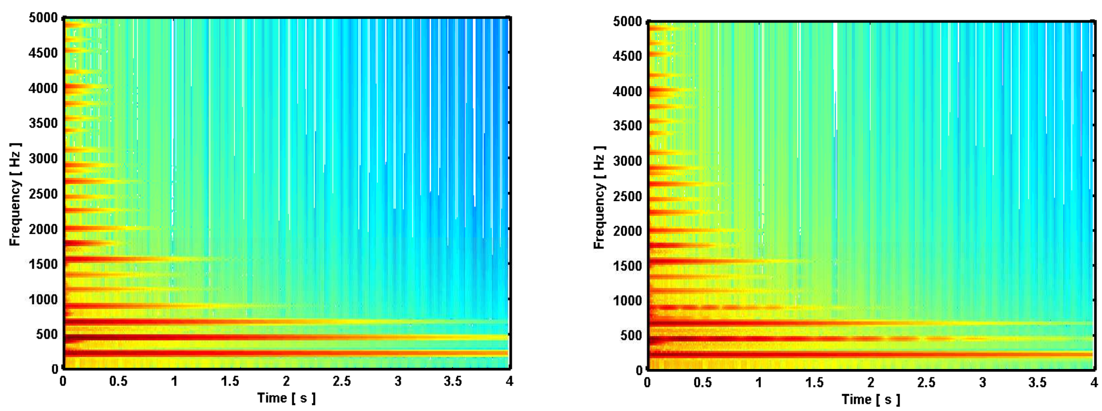

Figure 2: Spectrograms of the soundboard velocity computed from the linear string model. Left: one string simulation. Right: fully coupled model simulation.

3 Nonlinear coupled equations for the string dynamics

3.1 Equations for the three-dimensional string motions

As a general model, the three-dimensional coupled dynamics of a string at position x and time t, which involves an axial motion $X(x,t)$ and two perpendicular transverse motions $Y(x,t)$ and $Z(x,t)$, is given by the following set of equations, see for instance [12]:

$$\rho S \frac{\partial^2 X}{\partial t^2} - ES\frac{\partial^2 X}{\partial x^2} = (T_0 - ES)\frac{\partial}{\partial x}\left(\frac{1+\partial X/\partial x}{\|\partial \mathbf{R}/\partial x\|}\right) \tag{13a}$$

$$\rho S \frac{\partial^2 Y}{\partial t^2} - ES\frac{\partial^2 Y}{\partial x^2} = (T_0 - ES)\frac{\partial}{\partial x}\left(\frac{\partial Y/\partial x}{\|\partial \mathbf{R}/\partial x\|}\right) \tag{13b}$$

$$\rho S \frac{\partial^2 Z}{\partial t^2} - ES\frac{\partial^2 Z}{\partial x^2} = (T_0 - ES)\frac{\partial}{\partial x}\left(\frac{\partial Z/\partial x}{\|\partial \mathbf{R}/\partial x\|}\right) \tag{13c}$$

where

$$\|\partial \mathbf{R}/\partial x\| = \sqrt{\left(1+\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2} \tag{14}$$

describes the relative elongation of the string, ρ is the mass density of the string, S its cross-sectional area, E its modulus of elasticity and T_0 the initial tensioning axial force. As seen, only spatial derivatives of the string motions are present in the right hand-side forcing terms, and as such, nonlinear effects considered here come from purely geometrical effects. In the literature, Eqs. (13) and (14) are usually referred as the "geometrically exact" model of the vibrating strings. Although exact, the use of Eqs. (13) and (14) is rather limited in practice since no analytic solution can be found. Furthermore, from a numerical point of view, it does not allows separation between space and time-domain. To simplify the equations, different assumptions can be applied and a common practice is to use a Taylor expansion of Eq. (14) and then truncate the series with respect to the relative orders of the magnitude of the string displacements. Morse and Ingard proposed a simplified model by truncating the series up to the third order in the string displacements [12]. Interestingly, it highlights that mode-coupling arise from quadratic and cubic nonlinearities. A further but crude approximation is often made in the literature by ignoring the longitudinal-to-transverse coupling. For interested readers, a thorough treatment of approximate models stemming from the general model and often used for simulating the vibrating string has been given by Bilbao [5].

3.2 The Kirchoff-Carrier string model

3.2.1 Equations of the model

An alternative appealing model for the dynamics of nonlinear strings has been presented by Kirchoff [15] and Carrier [16]. Strickly speaking, it describes the transverse motions of the string and includes the effect of the nonlinearity as a driving term which is a function of the net increase in string length. The model thus encapsulates the changes in axial tension, but it is expressed globally in terms of a dynamical tension $T_{dyn}(t)$ computed by a spatial average of the square of the string slope as:

$$T_{dyn}(t) = \frac{ES}{2L}\int_0^L \left[\frac{\partial Y(x,t)}{\partial x}\right]^2 dx + \frac{ES}{2L}\int_0^L \left[\frac{\partial Z(x,t)}{\partial x}\right]^2 dx \tag{15}$$

The total tension of the string then reads:

$$T(t) = T_0 + T_{dyn}(t) \tag{16}$$

so that the three-dimensional dynamics of the string for large-amplitude is simply given as:

$$\rho S\frac{\partial^2 Y}{\partial t^2} = T_0\frac{\partial^2 Y}{\partial x^2} + \frac{ES}{2L}\frac{\partial^2 Y}{\partial x^2}\int_0^L \left[\left(\frac{\partial Y}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial x}\right)^2\right] dx \tag{17a}$$

$$\rho S\frac{\partial^2 Z}{\partial t^2} = T_0\frac{\partial^2 Z}{\partial x^2} + \frac{ES}{2L}\frac{\partial^2 Z}{\partial x^2}\int_0^L \left[\left(\frac{\partial Y}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial x}\right)^2\right] dx \tag{17b}$$

One recognizes the standard wave equations with an additional forcing term due to the distributed dynamical change in axial tension. However, by ignoring explictly the dynamics of the longitudinal motion, the validity of the Kirchoff-Carrier string model is rather restricted. In particular, one understands that all the dynamical responses of the longitudinal modes must be small. Such a situation occurs when the frequencies of excitation of the longitudinal modes are far lower than their resonant frequencies and according to Anand [1], this happens when the order of the excited transverse modes is smaller than the ratio of longitudinal to transverse wave propagation speed.

3.2.2 Modal formulation

The implementation of the Kirchoff-Carrier string model in the previsouly descrided modal computational model can be done in a straightforward manner. By looking at Eqs. (1) and (17), it is easy to see that the dynamical change of tension is reflected by an additional term in the modal force of the string transverse motions which are written, for both polarizations, as:

$$\mathcal{F}_n^Y(t) = T_{dyn}(t)\int_0^L \frac{\partial^2 Y}{\partial x^2}\phi_n^Y(x)dx = T_{dyn}(t)\sum_{m=1}^{N_Y} q_m^Y(t)\int_0^L \frac{\partial^2 \phi_m^Y}{\partial x^2}\phi_n^Y(x)dx \tag{18a}$$

$$\mathcal{F}_n^Z(t) = T_{dyn}(t)\int_0^L \frac{\partial^2 Z}{\partial x^2}\phi_n^Z(x)dx = T_{dyn}(t)\sum_{m=1}^{N_Z} q_m^Z(t)\int_0^L \frac{\partial^2 \phi_m^Z}{\partial x^2}\phi_n^Z(x)dx \tag{18b}$$

and which after performaing the integrations, takes the following form:

$$\mathcal{F}_n^Y(t) = -\frac{\pi^2}{2L}\,T_{dyn}(t)\,n^2\,q_n^Y(t) \qquad n = 1,\ldots,N_Y \tag{19a}$$

$$\mathcal{F}_n^Z(t) = -\frac{\pi^2}{2L}\,T_{dyn}(t)\,n^2\,q_n^Z(t) \qquad n = 1,\ldots,N_Z \tag{19b}$$

The modal form of the dynamical tension is then obtained by substituing the modal expansions (2) in (15) and by accounting for the orthogonality relationships of the string mode. After some calculations, it results in the following expression:

$$T_{dyn}(t) = \frac{ES\pi^2}{4L^2}\left(\sum_{n=1}^{N_Y} n^2\left[q_n^Y(t)\right]^2 + \sum_{n=1}^{N_Z} n^2\left[q_n^Z(t)\right]^2\right) \tag{20}$$

which, inserted in Eqs. (19), yields finally to:

$$\mathcal{F}_n^Y(t) = -\frac{ES\pi^4}{8L^3}\,n^2\left(\sum_{m=1}^{N_Y} m^2\left[q_m^Y(t)\right]^2 + \sum_{m=1}^{N_Z} m^2\left[q_m^Z(t)\right]^2\right)q_n^Y(t) \qquad n = 1,\ldots,N_Y \tag{21a}$$

$$\mathcal{F}_n^Z(t) = -\frac{ES\pi^4}{8L^3}\,n^2\left(\sum_{m=1}^{N_Y} m^2\left[q_m^Y(t)\right]^2 + \sum_{m=1}^{N_Z} m^2\left[q_m^Z(t)\right]^2\right)q_n^Z(t) \qquad n = 1,\ldots,N_Z \tag{21b}$$

The Kirchoff-Carrier model is thus of the third order in the amplitude of the tansverse motions. However, it is apparent that the model lacks some coupling terms regarding to the standard model originally derived by Morse and Ingard [12]. Indeed there is no quadratic coupling terms present in Eq. (21) and this limits in particular any comparison with precise measurements. Bank and Sujbert [6] discussed such limitations for applications to sound synthesis for the case of piano strings.

4 Time-domain simulations

Two cases have been chosen for illustration: the first simulation deals with a single string constrained at the point of the bridge with no coupling to the soundboard, and the second simulation pertains to the full 12-string model which includes the soundboard dynamics. The lowest string was puckled at a distance of one-fifth of the “acoustical length” of the string from the bridge, at an angle of 5 degrees to the z-plane. The strings total length is 0.615 m while their “acoustical length” is 0.44 m. The string were assumed harmonic and proportional damping assumptions were retained for both the strings and the body, using modal damping values of 0.01% and 1% for all modes of each subsystem respectively. Values of K_b = K_e = 10^5 N/m, and C_b = C_e= 1 Ns/m were used and for simplicity, the string/fret interaction was discarded by considering open strings. The time-domain numerical integration was performed by a standard explicit approach assuming a constant acceleration within the time-step [17]. The modal basis for the strings and soundboard cover the frequency range 0-5000 Hz, and time-domain responses were sampled using a time step of 2.10^{-7} s. Simulations starts by pulling a single string during 10 ms to reach an arbitrary position which is then released to vibrate freely. At the begining, all displacements and velocities are null. A total number of 607 modes is used.

	1^{st} pair	2^{nd} pair	3^{rd} pair	4^{th} pair	5^{th} pair	6^{th} pair
f_s [Hz]	493.88	440	329.63	493.88	440	293.66
	493.88	440	329.63	246.94	220	143.83
ρ_s [10^{-4} Kg/m]	3.78	3.94	6.20	3.78	3.94	11.30
	3.78	3.94	6.20	14.48	21.22	35.36
D [mm]	0.24	0.25	0.32	0.24	0.25	0.44
	0.24	0.25	0.32	0.5	0.64	0.79
(y_s,z_s) [mm]	(-28.6,16.0)	(-19.0,16.0)	(-9.3,16.0)	(0.0,16.0)	(9.5,16.0)	(19.8,16.0)
	(-25.5,16.0)	(-15.6,16.0)	(-6.2,16.0)	(3.1,16.0)	(12.8,16.0)	(23.3,16.0)
N_s	14	15	21	14/28	15/31	23/47

Table 1: Sounding frequency, linear density, modulus of elasticity, diameter, position and modal truncation order for the twelve strings.

Single string constrained at the bridge Figure 3 presents comparitive results obtained at varying pluck amplitude on the lowest string, with values of 0.001 m, 0.003 m and 0.01 m from up to bottom. It shows the temporal variations of the axial tension of the string as well as the spectrograms of the bridge/soundboard interaction force (only one foot is shown). For comparison, spectrograms were normalized to their energy contents before being processed. At low amplitude (see Fig. 3a), the total tension remains constant, close to the value of the initial tensionning of the string (T_0=59 N) and consequently, the behaviour of the string is essencially linear as attested by the straight lines in the spectrogram which represent the string modes. As the excitation amplitude increases, in Fig. 3b and 3c, one observes temporal variations of the string tension which can be as large as the static tension for large excitation amplitudes (Fig. 3c). As $T_{dyn}(t)$ becomes no longer negligible compared with T_0, nonlinear effects arise and this explains the striking frequency shift of the entire force spectrum observed in Fig. 3c. Also illustrated in the spectrograms is the enrichment of the force spectra with the excitation amplitude which is caused by the nonlinearity and has some clear audible effects.

Fully-coupled model Figure 4 shows two spectrograms of the bridge force stemming from the linear and nonlinear simulation model of the Portuguese guitar. It pertains to the pluck of the lowest string with an initial displacement of 10 mm in the $z-$direction. The linear model behaves as it should, with excited frequencies coming from a mix between strings and body modes. For the nonlinear simulation, a similar pitch gliding effect is observed as for the case of a simple string but one notices that some excited frequencies remain unchanged over the simulation. The corresponding horizontal lines represent the modes of the instrument body which responds linearly to the time-decreasing frequencies of the “string modes”.

5 Conclusions

This work aimed at continuing our efforts devoted to the complete modeling of the Portuguese guitar by using modal-based tecniques. By including the nonlinear behavior of the string for large-amplitude motions, the model provides a more precised description of the whole instrument dynamics. However, other improvements are still necessary and crucial issues includes the computations of both the coupled air-structure modes of the instrument body and the radiated sound. Nevertheless, at this point, our physical model pertains to the small group of studies which deals with the complete physical-based modeling of a real-life string instrument. The simulated nonlinear behaviour is clearly audible, similarly to what happens for real-life instruments.

Acknowledgments

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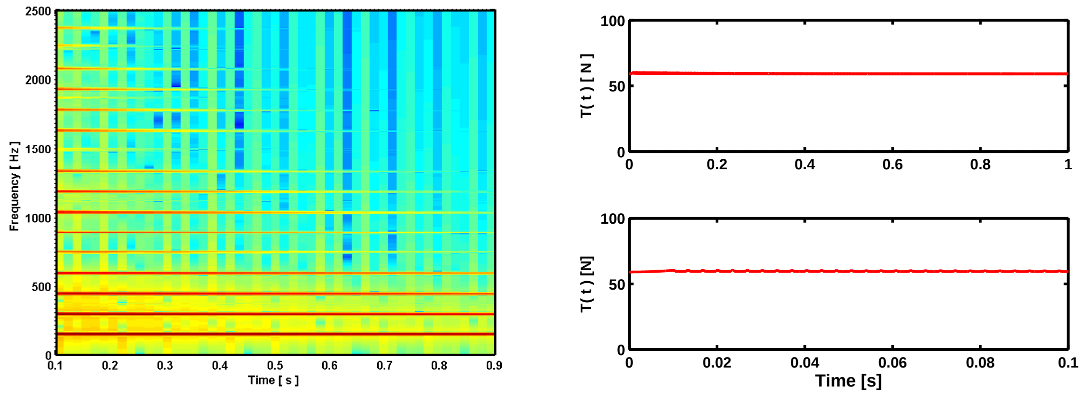
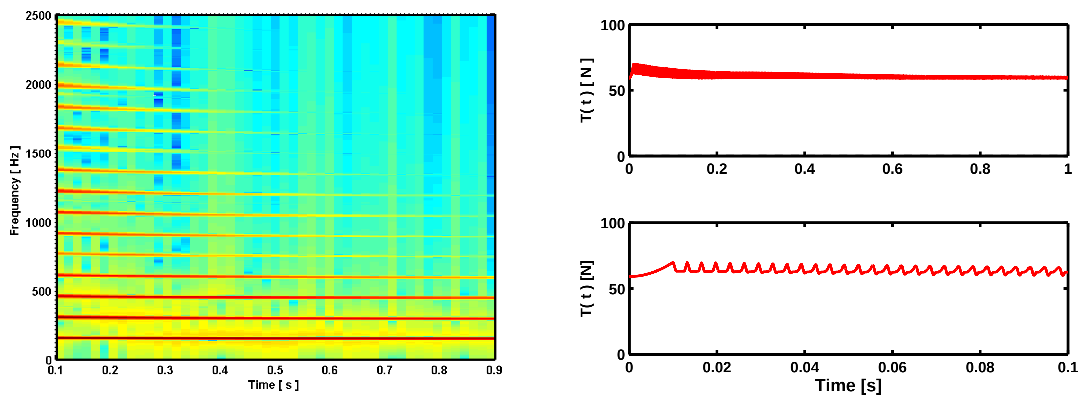
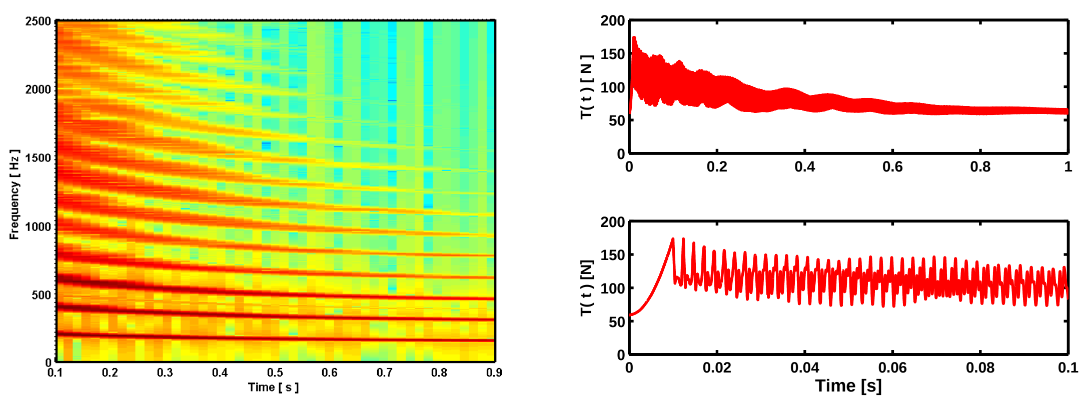


(a) Z_{max} = 0.001 m.

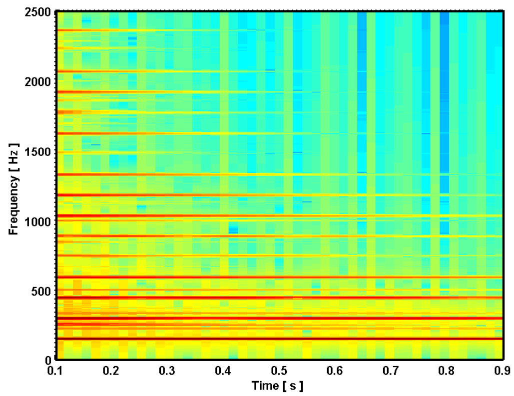


(b) Z_{max} = 0.003 m.



(c) Z_{max} = 0.01 m.

Figure 3: One string simulation. Spectrograms of the foot bridge force as a function of the amplitude of the pluck excitation.



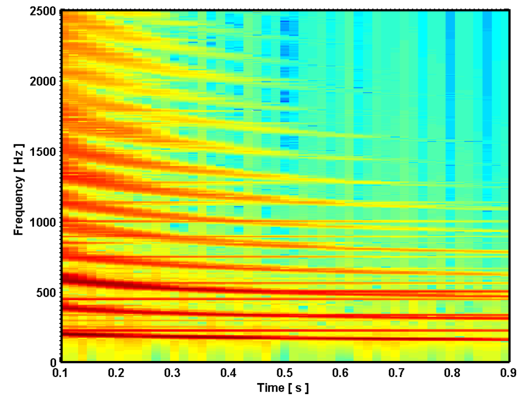


Figure 4: Fully-coupled model simulation. Spectrograms of the foot bridge force obtained by considering the linear (left) and nonlinear (right) formulation of the physical model.

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