ON MOMENTUM OF SOUND PULSES

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ABSTRACT
When a sound pulse reflects from a boundary with vacuum its momentum calculated according to classical expression changes sign. This apparently violates momentum conservation. It is demonstrated, that as a result of nonlinear interaction of the pulse with the boundary a second-order “massive” pulse (i.e., whose integral of density over the pulse extension is nonzero.) This pulse carries momentum which ensures overall momentum conservation, however does not carry energy (to the second order.)

INTRODUCTION
Classical result regarding momentum of a small-amplitude sound pulse reads:

\[ \vec{p} = \frac{E}{c_0} \bar{n} \]

where \( \vec{p} \) and \( E \) are momentum and energy of the pulse, \( \bar{n} \) is direction of its propagation, and \( c_0 \) is a sound speed [1]. However, there are situations where formal application of Eq. (1) leads to contradiction. Namely, consider sound pulse reflected from a boundary with vacuum. After reflection direction of the component of the momentum vector perpendicular to the boundary changes sign, and as a result momentum of the system seems to change without action of any external force. This paper is devoted to analysis of nonlinear effects associated with reflection of the pulse from the boundary with vacuum and resolution of this quandary.

FORMULATION OF THE PROBLEM
The equations describing sound propagation in homogeneous, barotropic medium read:

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + w(\rho) - w(\rho_0) = 0 \]

\[ \frac{\partial \rho}{\partial t} + \nabla (\rho \nabla \phi) = 0 \]

where \( \rho \) and \( \rho_0 \) are density and equilibrium density correspondingly, \( \phi \) is velocity potential, and \( w(\rho) \) is an enthalpy per unit mass

\[ w = \varepsilon + \frac{p}{\rho} \]

Where \( p = p(\rho) \) is pressure and \( \varepsilon = \varepsilon(\rho) \) is internal energy per unit mass. From Eqs. (2),(3) one easily obtains the momentum conservation law:

\[ \frac{\partial}{\partial t} \left( \rho \frac{\partial \phi}{\partial x_i} \right) + \frac{\partial}{\partial x_i} \left( \rho \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_k} + p \delta_{ik} \right) = 0 \]
Let us seek the solution of Eqs. (2),(3) in terms of power expansion with respect to small amplitude parameter:

\[
\rho = \rho_0 + \rho_1 + \rho_2 + \ldots, \quad \varphi = \varphi_1 + \varphi_2 + \ldots, \quad w = \frac{c_0^2}{\rho_0} \rho_1 + \alpha \rho_1^2 + \ldots
\]  

(6)

where constant coefficient \(a\) depends on the equation of state. At the boundary with vacuum the following condition holds: \(\rho = \rho_0\). Let us consider reflection of the 1D sound pulse propagating in homogeneous medium from the boundary with vacuum located initially at \(x = 0\). Position of the boundary \(\xi\) satisfies to the equation:

\[
\frac{\partial \xi}{\partial t} = \frac{\partial \varphi}{\partial x} \bigg|_{x=\xi}
\]

(7)

Integrating Eq. (5) with respect to \(x\) from \(-\infty\) to \(\xi\) one makes sure that the total momentum of the medium \(P\) is conserved:

\[
P = \int_{-\infty}^{\xi} \rho \frac{\partial \varphi}{\partial x} dx
\]

(8)

Substituting into this equation expansions Eq. (6) and assuming similar expansion for \(\varphi\) one finds to the accuracy of the second order terms:

\[
P = \rho_0 \varphi_2 \big|_{x=0} + \rho_1 \frac{\partial \varphi_1}{\partial x} \big|_{x=0} \cdot \xi_1 + \int_{-\infty}^{0} \rho_1 \frac{\partial \varphi_1}{\partial x} dx
\]

(9)

Let us assume that the sound pulse has small amplitude and consider solution of Eqs. (2),(3) up to the second order terms. In the linear approximation one finds:

\[
\varphi_1 = f \left(t - \frac{x}{c_0}\right) - f \left(t + \frac{x}{c_0}\right)
\]

\[
\rho_1 = -\frac{1}{c_0^2} f' \left(t - \frac{x}{c_0}\right) + \frac{1}{c_0^2} f' \left(t + \frac{x}{c_0}\right)
\]

\[
\xi_1 = -\frac{1}{c_0} f \left(t - \frac{x}{c_0}\right) - \frac{1}{c_0} f \left(t + \frac{x}{c_0}\right)
\]

(10)

where function \(f\) describes the incident pulse shape which is supposed to have a finite support. In the second approximation one obtains the following equations:

\[
\frac{\partial \varphi_2}{\partial t} + c_0^2 \frac{\varphi_2}{\rho_0} = -\frac{1}{2} \left(\frac{\partial \varphi_1}{\partial x}\right)^2 - \alpha \left(\frac{\rho_1}{\rho_0}\right)^2
\]

\[
\frac{\partial}{\partial t} \frac{\varphi_2}{\rho_0} + \frac{\partial^2 \varphi_1}{\partial x^2} = -\frac{\partial}{\partial x} \left(\frac{\rho_1}{\rho_0} \frac{\partial \varphi_1}{\partial x}\right)
\]

(11)

and boundary condition:

\[
\rho_2 \big|_{x=0} = -\frac{\partial \rho_1}{\partial x} \big|_{x=0} \cdot \xi_1
\]

(12)
If one neglects the first two terms in Eq. (9) which are related to boundary values one obtains:

\[
\int_{-\infty}^{0} \rho_1 \frac{\partial \varphi_1}{\partial x} dx = \frac{\rho_0}{c_0^2} \left\{ \int_{-\infty}^{0} \left[f'(t)\right]^2 dt, \quad t \to -\infty \right\} \left\{ \int_{-\infty}^{0} \left[f'(t)\right]^2 dt, \quad t \to +\infty \right\} (13)
\]

which indicates change of the momentum mentioned in the Introduction. Since the second term in Eq. (9) disappears at \( t \to \pm \infty \), balance of the momentum has to be due to the first term in this equation. Let us consider structure of the second order correction which ensures momentum conservation.

The second order terms appear due to both intrinsic nonlinearity of the process of propagation (which leads to generation of higher harmonics and eventually to formation of the shock wave) and to the interaction with the boundary. Let us consider solution of Eqs. (11),(12) with initial condition: \( \varphi_2 = \rho_2 = 0, \quad t = 0 \). If one assumes that the support of \( f \) is at positive argument then interaction with the boundary starts later at some positive moment of time. The second order solution can be explicitly calculated, however it is rather cumbersome. This solution can be represented as a superposition of terms of different structure most of which are nonzero only in the region where either incident or reflected pulse exists; they describe the effects of intrinsic nonlinearity of the pulse and interaction between the direct and reflected pulse:

\[
\varphi_2 = \sum \left\{ \int_{-\infty}^{x(t)} \left[f'(t)\right]^2 dt, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \right\} \left\{ \int_{-\infty}^{x(t)} \left[f'(t)\right]^2 dt, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \right\} (14)
\]

\[
\rho_2 = \sum \left\{ \int_{-\infty}^{x(t)} \left[f'(t)\right]^2 dt, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \right\} \left\{ \int_{-\infty}^{x(t)} \left[f'(t)\right]^2 dt, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \quad t \left[f(t)\right]^2, \right\} (15)
\]

In particular, these terms turn to zero at \( x = 0 \) for sufficiently large time. However, the first term in Eq. (14) calculated at \( x = 0 \) turns at \( t \to \infty \) to a constant value. This means that behind the reflected pulse appears a “tail” of a constant potential proportional to the square of the pulse amplitude. Let us calculate the value of the amplitude of this tail. Setting in the first of Eqs. (11) \( x = 0 \) and integrating with respect to time from \( -\infty \) to \( t \) one finds:

\[
\varphi_2 \mid_{t=0} = -\int_{-\infty}^{t} \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)_{x=0} \mid_{x=0} dt - c_0^2 \int_{-\infty}^{t} \frac{\rho_2}{\rho_0} \mid_{x=0} \mid_{x=0} dt \quad (16)
\]

and from Eq. (12) it follows:

\[
\frac{\rho_2}{\rho_0} \mid_{x=0} = \frac{\partial}{\partial x} \frac{\rho_0}{\rho_0} \mid_{x=0} \mid_{x=0} \mid_{x=0} \mid_{x=0} dt = \frac{4}{c_0^2} f''(t) f(t) \quad (17)
\]

Substituting Eqs. (10),(17) into Eq. (16) one finds:

\[
\varphi_2 \mid_{t=0, t=\infty} = -\frac{2}{c_0^2} \int_{-\infty}^{+\infty} \left[f'(t)\right]^2 dt - \frac{4}{c_0^2} f''(t) \mid_{-\infty}^{+\infty} + \frac{4}{c_0^2} \int_{-\infty}^{+\infty} \left[f'(t)\right]^2 dt = \frac{2}{c_0^2} \int_{-\infty}^{+\infty} \left[f'(t)\right]^2 dt \quad (18)
\]

Thus, among the second order terms generated as a result of interaction of the original pulse with the boundary there appears a left-propagating second order pulse:
\[
\tilde{\Phi}_2 = \tilde{f} \left( t + \frac{x}{c_0} \right)
\]  

(19)

Where function \( \tilde{f} \) has the following asymptotics:

\[
\tilde{f} = \begin{cases} 
\frac{2}{c_0^2} \int_{-\infty}^{+\infty} \left[ f'(t) \right]^2 dt, & t \to +\infty \\
0, & t \to -\infty 
\end{cases}
\]

(20)

This pulse carries the following momentum:

\[
\tilde{P} = \int_{-\infty}^{+\infty} \rho_0 \frac{\partial \tilde{\Phi}_2}{\partial x} dx = \rho_0 \tilde{\Phi}_2 \bigg|_{-\infty}^{+\infty} = 2 \rho_0 \int_{-\infty}^{+\infty} \left[ f'(t) \right]^2 dt
\]

(21)

Apparently, momentum of this pulse together with the momentum of the reflected pulse (the lower line in the RHS of Eq. (13)) equals to the momentum of the incident pulse (the upper line in the RHS of Eq. (13)).

Let us introduce "mass" of the pulse:

\[
\tilde{M} = \int_{-\infty}^{+\infty} \rho_0 \frac{\partial \tilde{\Phi}_2}{\partial x} dx = \rho_0 \tilde{\Phi}_2 \bigg|_{-\infty}^{+\infty} = 2 \rho_0 \int_{-\infty}^{+\infty} \left[ f'(t) \right]^2 dt
\]

(22)

which is in this case negative. Apparently, the following relation holds:

\[
\tilde{P} = -c_0 \tilde{M}
\]

(23)

The pulse propagates to the left with the velocity \(-c_0\), however its momentum is directed to the right because \(\tilde{M} < 0\). Appearance of such non-zero mass pulse can be anticipated due to mass conservation. As a result of interaction with the pulse the boundary experiences second order shift:

\[
\Delta \xi_2 = \xi_2 \bigg|_{t=+\infty}^{t=-\infty} = \int_{-\infty}^{+\infty} \frac{\partial \Phi_2}{\partial x} \bigg|_{t=0} dx + \int_{-\infty}^{+\infty} \frac{\partial^2 \Phi_1}{\partial x^2} \bigg|_{t=0} dx
\]

(24)

According to Eq. (10) the second term here turns to zero and contribution from the pulse under consideration is:

\[
\Delta \xi_2 = \int_{-\infty}^{+\infty} \frac{\partial \Phi_2}{\partial x} \bigg|_{t=0} dx = \frac{1}{c_0} \tilde{\Phi}_2 \bigg|_{t=+\infty}^{t=-\infty} = \frac{2}{c_0} \int_{-\infty}^{+\infty} \left[ f'(t) \right]^2 dt
\]

(25)

Shift of the boundary to the right leads to appearance of the pulse of depression with the corresponding mass:

\[
\tilde{M} = -\rho_0 \Delta \xi_2
\]

(26)

It is important to note, that energy of the pulse is at least a third order value; i.e. to the second order accuracy the pulse carries no energy, however carries non-zero momentum given by Eq. (21). For this reason appearance of this "massive" pulse does not violate energy conservation however ensures momentum conservation.
CONCLUSION

Derivation of the expression for sound pulse momentum Eq. (1) given in [1] carefully assumed that initial pulse in the terminology used above had zero mass and the medium was unbounded. However, in the presence of a boundary as a result of nonlinear interaction with the boundary “massive” pulses may appear which have non-zero integrals of density over the pulse length (a “mass” of the pulse.) These pulses carry momentum which equals to the product of the pulse mass and the speed of sound and depending on the sign of the mass can be directed either along- or opposite to the direction of the pulse propagation. Account of the momentum of these pulses ensures overall momentum conservation. However, these pulses to the second order accuracy carry no energy.

In the case considered above there is no dispersion, and both the reflected pulse and the second order “massive” pulse of depression will be always overlapped and travelling together. Let us assume that the original pulse is narrowband. Then in the presence of small dispersion these two pulses will eventually separate.

REFERENCES: