OPTICAL MEASUREMENT AND NUMERICAL SIMULATIONS OF THE SELF-MODULATED LOW FREQUENCY DISPLACEMENT

PACS: 43.25.-x

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ABSTRACT
Thanks to the ability of optical interferometers to measure accurately weak amplitude displacements of relatively low frequencies, we have studied the nonlinear self-demodulation of an ultrasonic tone burst propagating in a liquid, in the MHz range. Close to the source, the resulting low frequency displacement contains a quasi-static component, which is affected by diffraction effects farther from the transducer. The experimental set up provides quantitative results, which allowed us to determine the nonlinearity parameter of the liquid with a good accuracy. Such measurements were carried out both in water and ethanol. The pressure associated to this low frequency displacement contains both a standard cumulative contribution and a non-cumulative one. Introducing the temporal mean value of the displacement, the noncumulative part of the pressure is associated to the static part of the low frequency displacement. This interpretation leads to extend the definition of the Rayleigh radiation pressure usually limited to plane continuous wave radiated in a confined fluid. Results provided by a numerical simulation based on the KZ equation are in good agreement with experiments.

INTRODUCTION
Due to their complexity, optical sensors are rarely used to measure acoustic quantities. Nevertheless, they offer several advantages compared with standard piezoelectric hydrophones, like a better spatial resolution (typically 50 \(\mu m\), a good sensitivity and a larger bandwidth (especially in the low frequency domain). They constitute a reference method for the characterization of ultrasonic transducers and for the absolute calibration of piezoelectric hydrophones\textsuperscript{1,2}. In this paper, the ability of optical interferometers to measure accurately weak amplitude displacements of relatively low frequencies is used to study the nonlinear self-demodulation of an ultrasonic tone burst in the MHz range.

"Self-demodulation" refers to the low-frequency signal, which is nonlinearly generated by the propagation of a higher frequency tone burst. It results from the nonlinear interaction of the harmonic components contained in the tone burst spectrum. This LF acoustic pressure has been already studied experimentally and numerically in various media\textsuperscript{3,4,5}. In this paper, according to the quantity measured by the optical interferometer, we work on the displacement \(U\) of a fluid particle referenced by its position \(a\) at rest. Using Lagrangian coordinates \((a, t)\), let us so consider the nonlinear propagation equation for a plane wave, in a semi-infinite \((a > 0)\) lossless fluid\textsuperscript{12}:

\[
\frac{1}{c_0} \frac{\partial U}{\partial t} + \frac{\partial U}{\partial a} = \frac{\beta}{2} \left( \frac{\partial U}{\partial a} \right)^2. \tag{1}
\]

\(c_0\) is the sound speed of the fluid at rest, and \(\beta\) is the acoustic nonlinearity parameter. Since we focus on the nonlinear propagation of a tone burst, the transient source condition is: \(U'(a=0,t) = U_0(t) \sin(\omega_0 t)\), where \(U_0(t)\) is the amplitude modulation, which slowly varies versus time in comparison with the term of angular frequency \(\omega_0\). In order to establish an approximate solution of Eq. (1), we use the method of successive approximations: \(U = U_1 + U_2 + ..., \) where \(U_2 \ll U_1\). In this "quasi-linear" approach, \(U_1\) denotes the displacement in linear regime (first order in acoustic Mach number \(M = \omega_0 U_0/c_0\)) and \(U_2\) represents the waves induced by the nonlinear propagation (second order in \(M\)). Introducing the retarded time \(\tau = t - a/c_0\), assuming that the primary wave \(U_1\) propagates without any change of its waveform (so that
\[ \frac{\partial U_0(t)}{\partial a} = 0 \] and omitting the nonlinear term corresponding to the second harmonic, Eq. (1) becomes at second order in \( M \):

\[
\left( \frac{\partial U_2}{\partial a} \right)_a = \frac{\beta a_0^2}{4c_0^2} U_0^2(t). \tag{Eq. 2}
\]

The low-frequency (LF) displacement of the fluid particle located at a distance \( z \) from the source, is deduced from Eq. 2:

\[
U_{LF}(a = z, t) = \frac{\beta}{4} k_0^2 z U_0^2 \left( t - \frac{z}{c_0} \right), \tag{Eq. 3}
\]

where \( k_0 = \omega c_0 / c_0 \) is the wave number. Increasing linearly with the acoustic intensity and with the propagation distance \( z \), this LF displacement results clearly from a nonlinear and cumulative process. For a high frequency continuous wave, the displacement \( U_{LF} \) is static and it is proportional to the square of the amplitude \( U_0 \) of the emitted displacement.

In a real fluid, where thermoviscous phenomena have to be taken into account, the LF displacement \( U_{LF} \) and the absorption coefficient \( a_0 \) of the primary wave both increase as the square of the frequency. So this frequency must be appropriately chosen in order to lead to a sufficiently large LF displacement, without any prohibitive absorption limiting dramatically the nonlinear process. For example, in water (\( c_0 = 1500 \text{ m/s}, \beta = 3.5 \)) and for a primary wave amplitude \( U_0 = 5 \text{ nm} \) at a frequency \( f_0 = 10 \text{ MHz} \) (\( a_0 = 2.4 \text{ Np/m} \)), a 1 nm LF displacement would be expected at a distance \( z = 25 \text{ mm} \) from the transducer. At this distance, the primary wave attenuation is only 0.5 dB.

In this paper, we show that such small LF displacements can be measured with a sensitive optical interferometer. Measurements are carried out in a large water tank at different distances from the transducer and for various amplitudes of a tone burst whose temporal shape is rectangular. We verify experimentally that, close to the source, the LF displacement contains a quasi-static component, which is affected by diffraction effects farther from the transducer. If diffraction effects do not invalidate Eq. 3, the acoustic nonlinearity parameter of the fluid can be measured. Experimental results are in good agreement with those provided by a numerical simulation based on the KZ equation. Finally, we discuss about the LF pressure \( P_{LF} \) associated to \( U_{LF} \). This pressure contains both a standard cumulative contribution and a non-cumulative one. Concerning this latter, we suggest that close to the source, a quasi-static pressure is associated to the quasi-static part of the LF displacement. It may be identified as an extension of the Rayleigh radiation pressure usually introduced for plane continuous waves radiated in a confined fluid. Because of the diffraction of the LF displacement, this quasi-static pressure is cancelled when the propagation distance increases.

**EXPERIMENTS AND NUMERICAL SIMULATIONS**

An optical interferometer is used to measure the LF displacement nonlinearly induced by an ultrasonic tone burst having a rectangular shape. Experimental results are compared with those given by a numerical simulation.

**Experimental set up**

The experimental set up is represented in Fig. 1. A piezoelectric transducer (central frequency \( f_0 = 10 \text{ MHz} \), diameter \( d = 10 \text{ mm} \)) is immersed in a water tank whose size is very much larger than the acoustic beam. This transducer is mounted on a micrometric stage allowing displacements along three perpendicular axes. It radiates a tone burst, which is obtained by multiplying a harmonic signal with an adjustable duration rectangular envelope. The duration is chosen short enough (1 \( \mu \text{s} \)) to lead to a quasi-transient diffraction regime, allowing the assumption of a plane wave in the near field area. The amplitude of the transmitted ultrasonic wave is controlled via an attenuator. Before its application to the transducer, the electric signal is band-pass filtered, in order to eliminate the possible components nonlinearly generated by the emission device.

Wave induced displacements are measured by a high sensitive optical heterodyne interferometer, whose probe beam is reflected by a thin metallized membrane of Mylar (thickness 12 \( \mu \text{m} \)) immersed in front of the transducer. As already described elsewhere, the modulation of the path of the probe beam is transposed into a phase modulation \( \Delta \Phi \) of the photodiode current in the radiofrequency domain by a change of the optical frequency with an
acousto-optic modulator (Bragg cell operating at the frequency \(f_0 = 70\) MHz). Then, the normal displacement \(U\) of the membrane modulates the optical beam phase, according to the relation:
\[
\Delta \Phi(t) = 2K_{\text{eff}} U(t),
\]
where \(K_{\text{eff}} = 2\pi n_{\text{eff}} \Lambda\) is the effective optical wave number, taking into account the modification of the optical refraction index of water induced by the acousto-optic interaction. For a plane wave propagating in water, it can be shown\(^5\) that \(n_{\text{eff}} = 1\).

![Diagram of experimental setup](image)

**Figure 1.**-Experimental set up

This phase modulation can be numerically or analogically extracted from the photodiode current. In order to reduce electronic noise, the photocurrent is band-pass filtered between \(f_0 \pm 25\) MHz. The LF displacement \(U_{LF}\), corresponding to the low frequency components of the phase modulation, is selected by using a narrower bandpass filter \((f_0 \pm 3\) MHz).

This experimental set up leads to the displacement measurement of a fixed fluid particle. Moreover, taking into account the thickness of the membrane (12 \(\mu\)m) and the impedance of Mylar (2.9 MRayl.), the membrane can be considered as transparent to ultrasounds in this frequency range. This transparency ensures the absence of a reflected wave. Then, the measured displacements are Lagrangian quantities.

**Numerical simulation**

A numerical simulation, based on a finite different scheme of the 2D KZ equation, has been used to simulate our experiment. This simulation has been already fully described elsewhere\(^5\). It is based on a potential formulation of the generalized KZ equation, which improves the treatment of the shock waves. Moreover, this formulation is well suited for our experiment, because the potential is directly proportional to the displacement measured by the optical interferometer. The numerical algorithm uses a split step procedure to treat separately on one-hand diffraction and attenuation in the frequency domain, and on the other hand nonlinearities in the time domain. The imposed “input” boundary condition corresponds to the displacement field measured at the source, i.e. very close to the transducer (at a distance inferior to 0.5 mm).

**Experimental and numerical results**

![Space-time representations](image)

**Figure 2.** - Space-time representations of the tone burst displacement \(U\) and associated LF component \(U_{LF}\), at a distance \(z = 3\) mm. (a) Experiment, (b) Numerical results.

First experiments were carried out at 10 MHz with a planar transducer having a 10 mm diameter. In Fig. 2-a, space-time representations of both tone burst and LF displacements are reported for a distance from the transducer to the membrane equal to \(z = 3\) mm. The LF
displacement component has a rectangular temporal shape similar to the tone burst envelope and an amplitude smaller than 1 nm. When placing the membrane as close as possible to the transmitter, no LF component is detected. Thus, this component is not directly radiated by the transducer. The spatial extension of the LF frequency displacement corresponds to the transducer diameter. In Fig. 2-b, results given by the numerical simulation are in good agreement with experiments.

In Fig. 3, measured and simulated displacements are represented as a function of time, for \( z = 3 \) mm, for various amplitudes of emission. We find a rectangular LF displacement shape. The LF displacement amplitude varies quadratically with the tone burst amplitude (see also Fig. 5-b). These results are similar to those recently published in the case of a solid\(^6\).

![Figure 3](image1)

Figure 3.- Displacements (tone burst \( U \) and LF component \( U_{LF} \)) at a distance \( z = 3 \) mm from the source, for various excitation amplitudes. The tone burst displacement corresponds to the highest level amplitude. (a) Experiments, (b) Comparison of experimental (—) and numerical (— —) results.

At a distance \( z = 22 \) mm, the temporal shape of the LF displacement changes (Fig. 4). It tends towards the time derivative of the wave form observed close to the source. In linear acoustics, this is a characteristic of a transient diffraction regime: in the far field, the temporal waveform tends towards two short pulses with opposite signs. The waveforms obtained with the numerical simulation differ a little from experimental results. These discrepancies can be ascribed to the 2D treatment of diffraction in the numerical simulation.

![Figure 4](image2)

Figure 4.- Displacements (tone burst \( U \) and LF component \( U_{LF} \)) at a distance \( z = 22 \) mm from the source, measured for various excitation amplitudes. The tone burst displacement corresponds to the highest level amplitude. (a) Experiment, (b) Comparison of experimental (—) and numerical (— —) results.

Fig. 5-a shows the measure of the LF displacement amplitude versus the distance \( z \) between the planar transducer and the membrane. The amplitude increases linearly until a distance \( z = 25 \) mm, from which the cumulative process is limited by the diffraction of the LF component. These measurements have been performed using \( a \theta = 1 \) μs duration envelope. So the Fresnel diffraction distance \( d^2/(8c_v\theta) \), calculated as if \( U_{LF} \) was emitted in linear regime by the transducer, is about 8 mm. Increasing \( T \), we have checked experimentally that the diffraction distance of \( U_{LF} \) decreases. However, in this experiment, the interference of the main and edge LF waves occurs at a larger distance (\( z \approx 20 \) mm). Thus, in spite of the linear diffraction effect, the LF displacement component remains collimated farther, thanks to the nonlinear cumulative antenna effect. The attenuation and Fresnel distances of the high frequency tone burst are respectively: \( L_a = 1/c_v = 42 \) cm and \( d^2c_v/(4c_v\theta) = 16.7 \) cm. So neither the attenuation nor the diffraction of the primary beam limits the cumulative process.
The nonlinearity parameter of the fluid can be determined from Eq. 3. For this purpose, the LF displacement amplitude is measured for various emission amplitudes \( U_0 \) at a distance \( z \) where it is not perturbed by diffraction effects. The LF displacement amplitude is represented in Fig. 5-b, as a function of the quantity \( x = zk^2U_0^2 / 4 \) (in nanometre). A linear dependence is clearly observed, which confirms the quadratic behaviour of \( U_L \) versus \( U_0 \). According to Eq. 3, the slope value, deduced from a least mean squared linear regression, is equal to the fluid nonlinearity parameter \( \beta \). Values obtained for water (\( \beta = 3.57 \)), and ethanol (\( \beta = 6.42 \)) are in good agreement with those found in the literature (\( \beta_{\text{water}} = 3.5 \) and \( \beta_{\text{ethanol}} = 6.15 \))\(^{10} \). Thus, the self modulated signal can provide another method to measure nonlinearity parameters, alternative to usual ones\(^{11, 12, 13} \).

Figure 5.- (a) Measurement of the LF displacement amplitude \( U_L \) versus the propagation distance \( z \). (b) Nonlinearity parameter \( \beta \) measurement in water (\( z = 20 \, \text{mm} \)).

DISCUSSION ABOUT THE ASSOCIATED PRESSURE

In this section, we discuss about the pressure associated to the LF displacement experimentally studied. Let us first consider the second order state equation in a lossless fluid:

\[
P - P_0 = A \left( \frac{\rho - \rho_0}{\rho_0} \right) + B \left( \frac{\rho - \rho_0}{\rho_0} \right)^2,
\]

(Eq. 4)

where \( P \) and \( \rho \) are the fluid pressure and density, \( P_0 \) and \( \rho_0 \) are their values at rest and \( A = \rho_0 c_0^2 \) and \( B = \rho_0^2 \left( \frac{\alpha^2 P/\varphi^2}{\rho_0} \right) \) are respectively the first and second order coefficients of the Taylor expansion. Using the 1D-continuity equation \( (\rho_0 / \rho) = 1 + \partial U / \partial a \), Eq. 4 becomes:

\[
P - P_0 = -A \frac{\partial U}{\partial a} + \beta A \left( \frac{\partial U}{\partial a} \right)^2.
\]

(Eq. 5)

Following the perturbation method described in the introduction, the second order acoustic pressure in \( M \) is: \( p_2 = -A (\partial U_j / \partial a) + \beta A (\partial U_j / \partial a)^2 \). Since we focus only on the LF components of \( p_2 \), Eq. 3 is used to calculate the second order displacement \( U \). The LF pressure is then:

\[
p_{LF} (z, \tau) = A \beta \kappa_0^2 \frac{z}{c_0} \frac{\partial U_0^2 (\tau)}{\partial \tau} + U_0^2 (\tau).
\]

(Eq. 6)

\( p_{LF} \) contains both a cumulative contribution (proportional to the propagation distance \( z \)) and a non-cumulative one. Following the weak nonlinearity approximation, the superposition principle still holds for nonlinearly induced waves and these two contributions to the pressure add.

The cumulative contribution to \( p_{LF} \) results from the nonlinear interaction of the different frequency components contained in the spectrum of the emitted tone burst. It has been fully studied in literature, where measurements have been realized using hydrophones\(^4 \). In our experiment, close to the source, this contribution to the LF pressure would be zero between the edges of the rectangular envelope of the tone burst.

In a recent paper\(^{14} \), we have proposed, as already done in the case of lossless solids\(^{15} \), to consider the constant part of the LF displacement in the near field of the transducer, as a non-zero temporal mean value of the corresponding part of the total displacement. Using this approach, the quasi-static part of the LF displacement measured in the near field of our transducer can be related to a quasi-static pressure. This interpretation leads to extend the definition of the Rayleigh radiation pressure, which usually concerns continuous plane waves in confined fluids\(^ {16, 17} \). In our experiment, the transducer is sufficiently directive to confine the nonlinearly induced quasi-static pressure in its near field. Farther from the source, the diffraction
effect affecting $U_{LF}$ can be regarded as the equalization of this quasi-static pressure with the ambient pressure $P_0$ in the surrounding fluid. The self-demodulated displacement can also be linked to this quasi-static pressure, but it requires the tone burst to reach a steady state regime (i.e. to have a sufficiently large rectangular envelope, compared with $T_0$).

The order of magnitude of the cumulative contribution to $p_{LF}$ is ten times greater than the quasi-static pressure one\textsuperscript{14}, for a propagation distance $z = 3$ mm. Moreover, as it is cumulative, it increases with the propagation distance, whereas diffraction cancels the quasi-static pressure. However, even if the cumulative part of the acoustic pressure dominates, the two contributions do not appear at the same time in the signal.

CONCLUSIONS

This study deals about the self-modulation of a 10 MHz frequency tone burst, having a rectangular envelope with a relatively short duration. Using an optical interferometer coupled with a metallized Mylar membrane immersed in the fluid, measurements have been carried out for various emission amplitudes, at different distances from the source. In the near field, the self-modulation produces a LF displacement having a rectangular shape like the tone burst envelope. The acoustic nonlinearity parameter deduced from the LF displacement measurements is in good agreement with previous results. Farther from the source, the increase of the LF displacement amplitude with the propagation distance is limited by the effect of diffraction. The pressure associated to the LF displacement was discussed under a plane wave assumption. There are two contributions. A first one, well known, is cumulative and results from the nonlinear interaction of the different frequency components of the tone burst. A second one, non-cumulative, results from a constant mean displacement gradient. This quasi-static pressure associated with the static part of the LF displacement has been formally identified as the Rayleigh radiation pressure, which is here extended to the case of a tone burst in an unconfined fluid. Experimental results are in good agreement with those provided by a numerical simulation based on the KZ equation. In the future, we plan to use this simulation to investigate the effect of the Rayleigh radiation pressure on a solid particle immersed in the fluid.


19TH INTERNATIONAL CONGRESS ON ACOUSTICS – ICA2007/MADRID