COMPLEX DYNAMICS IN A MAGNETOACOUSTIC RESONATOR

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ABSTRACT
The parametric generation of ultrasound resulting from magnetoelastic interactions in a ferromagnetic material driven by an alternating magnetic field is considered. We present a dynamical model taking into account both purely magnetic and magnetoacoustic nonlinearities, which describes the response to a non-resonant (detuned) excitation. The stability of the stationary solutions is analyzed. The analysis reveals the existence of self-oscillatory solutions. Near the bifurcation point, the response is quasi-harmonic. Decreasing the driving a more complex scenario is observed. First, ultrasound is emitted in the form of bursts of high amplitude with a large separation in time. This behaviour occurs when the pumping magnetic field value is above but near the threshold of parametric generation. Below the threshold, we observe a dynamical behaviour which is called excitability: weak perturbations of the equilibrium (subharmonic off) state, but above a given threshold value, are highly amplified, and the medium emits an acoustic pulse which relaxes to the equilibrium state. Our numerical analysis of the model show these and other features, demonstrating the excitable character of the system. Other regimes, including chaotic response, have been identified for different sets of parameters.

INTRODUCTION
Magnetostriction is essentially a nonlinear phenomenon, which accounts for the change of dimension of a magnetic material under applied magnetic fields. However, most of the applications of this phenomenon consider small amplitude fields and consequently exploit only their linear properties, as is the case of electromagnetic-acoustic transducers. The study of parametric generation of ultrasound by alternating magnetic fields dates from several decades ago (see, e.g., [1] and references therein). More recently, the consideration of nonlinearities in the description of this process lead to the discovery of new properties, such as the parametric wave phase conjugation (WPC), a phenomena which is actually an active field of research with promising applications into acoustic microscopy [2] or harmonic imaging [3]. A review on WPC theory and methods can be found in [4]. There is also increasing interest in the development of magnetostrictive transducers working at high powers, where nonlinear effects are not negligible. The advances in this field come in parallel with the search of novel magnetic materials with high magnetostriction values.

On the other side, parametric phenomena in different fields of nonlinear science share many common features, such as bistability, self-pulsations and chaos among others. Is precisely this analogy that motivates our search for complex dynamical phenomena in parametrically driven magnetoacoustic systems.

Different models have been proposed for the description of magnetoacoustic interaction in ferromagnetic materials. Also, considerable experimental progress in this field has been achieved, and the main parameters involved in the process have been determined [5]. In a previous paper [6] we analyze, in the resonant case, the dynamical behaviour of this system in the presence of magnetic nonlinearity. It was shown that a cubic magnetic nonlinearity is responsible for the appearance of new effects not reported before in this system, such as self-pulsing dynamics and spiking behaviour, related to the existence of homoclinic bifurcations [7]. In this work we extend the analysis considering a detuned operation, describing a more realistic situation where the input frequency can be varied.
THE MODEL

The system under study is a bar or disk of magnetostrictive material (e.g. hematite) where the plane and parallel surfaces form an acoustic resonator for the parametrically generated ultrasound. The driving is provided by an oscillating magnetic field produced by a coil with \( n \) turns surrounding the magnetic material. The coil provides the inductance of an electric series RLC circuit, driven by the external ac source at frequency \( 2\omega \) variable amplitude \( \varepsilon \).

Owing to magnetoelastic coupling, elastic deformations in the magnetostrictive material result in an additional magnetic field. Parametric generation implies the excitation of an acoustic mode at half of the frequency of the driving, under a periodic modulation of one of the system parameters (e.g. the sound velocity). This implies that the dominant contribution of the magnetoelastic interaction is quadratic in the particle displacements, \( H_{\text{int}}(t) = \alpha <u(r,t)^2> \), where \( \alpha = (2k)^2 \partial \ln \sqrt{v} / \partial H \) is the coupling coefficient proportional to the modulation depth of sound velocity [8], and the brackets indicate a spatial average over the material volume.

Taking into account the magnetoelastic contribution, the effective magnetic excitation in the material takes the form \( H = H_{\text{ext}}(t) + H_{\text{int}}(t) \), where the external field has both stationary and oscillating components directed along the axis of the resonator, i.e. \( H_{\text{ext}}(t) = H_0 + H_q(t) \). The resulting magnetic induction \( B = \mu (H) H \), is in general a nonlinear relation which to the leading order can be written as \( B = \mu H + \mu_0 \chi^{(3)} H^2 \) [9], where \( \mu \) is linear the permeability of the material and \( \chi^{(3)} \) the third order magnetic susceptibility, which in turn depends on the frequency.

A nonlinear equation for the circuit can be obtained under the previous assumptions, and neglecting no resonant terms and those higher than quadratic in \( H_q \) and \( H_{\text{tot}} \). It reads

\[
L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = e \cos(2\omega t) + \mu_n a \frac{d}{dt} \left[ u^2 dV + \mu_0 \chi^{(3)} n^2 H_q \frac{d}{dt} \left( \frac{dq}{dt} \right)^2 dV \right]
\]

where \( q \) is the charge in the capacitor, related with the current as \( I = dq/dt \), \( L \) is the coil inductance and \( H_q = nI \). The last two terms result from the nonlinearities related to magnetoelastic interaction and magnetic nonlinearity.

The acoustic field obeys the wave equation with a source (coupling) term proportional to the magnetic field (8). In terms of the charge it reads

\[
\frac{L}{v^2} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = a Hu = an \frac{dq}{dt} u .
\]

We consider the quasi-harmonic (slowly varying amplitudes) solutions of of Eqs. (1) and (2), in the form \( q(t) = Q(t) \exp(2i\omega t) + c.c. \) and \( u(r,t) = [U(t) \exp(i\omega t) + c.c.] \sin(kz) \), where \( k = m\pi / l \) correspond to a cavity eigenmode, and \( 2\omega \) is the forcing frequency. The forcing is detuned (out of resonance) if \( 2\omega \) is not coincident with the natural frequency of the circuit, \( \omega_c = 1 / \sqrt{LC} \). In this case, the parametrically generated ultrasound at frequency \( \omega \), half of the driving frequency, will be in general also detuned with respect to a cavity eigenfrequency \( \omega'_c = mc\pi / l \). Following the procedure described in detail in [6], the following dynamical system is obtained for the slow amplitudes.
\[ \frac{dX}{dt} = P - (1 + i \delta_x) X + Y^2 + i \gamma X Y \]
\[ \frac{1}{\gamma} \frac{dY}{dt} = (1 + i \delta_y) Y + XY^* \] 

(3)

where \(X\) and \(Y\) are proportional to the slow amplitudes of the charge in the capacitor and the ultrasonic field, respectively, and \(P\) is proportional to the input voltage (see [6,7] for definitions). The parameter \(\gamma\) represents the ratio between acoustic and electric losses, denoted by \(\gamma_x\) and \(\gamma_y\) respectively. The detunings are defined as \(\delta_x = (2\omega - \omega_e) / \gamma_x\) and \(\delta_y = (\omega - \omega_n) / \gamma_y\), where \(\omega_n\) is the cavity mode closest to \(\omega\). The dimensionless time \(\tau = \gamma_q t\) has been also defined.

Finally, \(\eta = 4 \omega_H \gamma_x X^{(3)} / \omega \gamma^2 \mu_r\) is the nonlinearity parameter. One can reduce the number of parameters by imposing that the electrical resonance has the value corresponding to an acoustic cavity mode. In such a case, the normalized detunings in Eqs. (3) obey \(\delta_x = \delta_y / 2\gamma = \delta\).

**STATIONARY SOLUTIONS AND STABILITY**

Equations (3) possesses two kinds of stationary solutions. For small pump values (below threshold), the acoustic subharmonic field is absent, and the homogeneous trivial solution is readily found as

\[ |X| = \frac{P}{\sqrt{1 + \delta^2}}, \quad |Y| = 0 \] 

(4)

For higher values of the pump, the trivial solution becomes unstable and the acoustic subharmonic field is switched on. The corresponding values of the amplitudes are given by

\[ |X|^2 = 1 + \frac{d^2}{4\gamma} \] 
\[ |Y|^2 = \frac{4\gamma^2 (1 - d\eta) - 2\eta d^2 - 2d^2 \pm \sqrt{4\gamma^2 P^2 (4\gamma^2 \eta + d^2 \eta^2 + 4\gamma^2 (1 + \eta^2)) - 2\eta d + \delta^2 + 4\gamma^2 (\delta + \eta)^2}}{4\gamma^2 (1 + \eta^2) + d^2 \eta^2 + 4\gamma^2 \eta} \]

the bifurcation occurs at a critical (threshold) pump given by

\[ P_{th} = \frac{1}{2\gamma} \sqrt{\left(1 + \delta^2 \right)(4\gamma^2 + d^2)} \] 

(6)

The acoustic field shows bistability, i.e., both solutions coexist, for pump values between \(P_b\) and \(P_{th}\), where \(P_b\) corresponds to the turning point of the solution.

In Fig.1 the intensity of the generated ultrasound is plotted versus the pump value in two cases: the resonant (\(\delta = 0\)) and detuned (\(\delta = 2\)) cases. The bifurcation points as evaluated from the linear stability analysis denoted with squares. Note that the presence of detuning increases the value of the parametric threshold, and affects the form of the solutions and the location of the instability points.
We have also studied the stability of the stationary solutions (Eqs.5), by means of a linear stability analysis. Substituting in Eqs. (Eqs.3) and their complex conjugate a perturbed solution in the form \( x_i(t) = \bar{x}_i + \delta x_i(t) \), where \( \bar{x}_i \) is a vector with the particular stationary values, and linearizing the resulting equations around the small perturbations, one obtains that \( \delta x_i \sim e^{\lambda t} \), where \( \lambda \) are the growth rates (eigenvalues of the stability matrix) whose real parts indicate the presence of instabilities. Since we are interested in the existence of dynamical behaviour, we search for a pair of complex conjugate eigenvalues, which denote the occurrence of a Hopf (oscillatory) instability. In this case, the eigenvalue reads \( \lambda = \pm i\omega \), where \( \omega \) is the angular frequency of the self-oscillations. The expressions, although analytical, is quite involved and in the resonant case it can be found in [6]. In Fig. 1 the location of the bifurcations for two cases is marked with squares (HP), the solution being unstable (oscillatory) for pump values below these points.

Figure 2 (left) shows the results of the numerical simulation of Eqs. (3) for parameter values near the Hopf bifurcation. The resonant case was considered, although a similar behaviour was observed in the presence of detunings.

**FIGURE 2.** Temporal evolution of the acoustic intensity, resonant case. The value of nonlinearity coefficient was \( \eta = 0.7 \), and the losses \( \gamma = 0.1 \). Quasi-harmonic self-pulsing near the Hopf point (left), for \( P = 1.4 \) and spiking oscillation near the homoclinic point (right), for \( P = 0.95 \).

**HOMOCLINIC DYNAMICS AND EXCITABILITY**
In [7] it was also shown that the system possesses homoclinic (global) dynamics at a particular value of the nonlinearity parameter $\eta$. In this letter we perform a detailed two-parameter numerical analysis of Eqs. (3). For this aim, we reduce the dimensionality of the parameter space by fixing the value of the relative losses $\gamma = 0.1$, which is in correspondence with typical experimental conditions. The results for other values of $\gamma$ are qualitatively similar.

The bifurcation diagram shown in Fig. 3 (left) reveals that the system presents a complex scenario of global dynamics. Besides the Hopf bifurcation (HB) leading to self-pulsing, three different homoclinic bifurcations have been identified in this system. These bifurcations can be detected numerically by computing the period of the limit cycles, since close to a homoclinic/heteroclinic point the period of oscillations diverges to infinity as $T = -(1/\lambda)(P_h - P)$, where $P_h - P$ measures the distance to the homoclinic bifurcation (which is assumed small) and $\lambda$ is the eigenvalue in the unstable direction of the saddle point. In Fig. 3, infinite period ($\infty$) bifurcations are the curves labeled (a)-(c). Curve (a) corresponds to a gluing (double homoclinic) bifurcation. This bifurcation is characteristic of systems with $Z_2$ symmetry and is mediated by a saddle point, which in this case corresponds to the trivial state. The gluing bifurcation exists for a broad range of pump values, and persists until the pitchfork bifurcation, at $P = 1$. At this point, two new branches of $\infty$ bifurcations emerge. The upper branch (b) correspond to a homoclinic bifurcation connecting one saddle with itself, while the lower branch (c) corresponds to a heteroclinic connection between two symmetric saddles.

In the bistability there exist a domain (dark-shadowed in Fig. 3) where the system is excitable: weak perturbations of the equilibrium (subharmonic off) state, but above a given threshold value, are highly amplified, and the medium emits an acoustic pulse which relaxes to the equilibrium state. On the other side, the amplitude of the response is independent of magnitude of the perturbation. This behaviour is typical of some (few) dynamical systems, the most representative being the neuron. In Fig. 3 (right) we show the system’s response to perturbations of increasing amplitude, in the excitable region.

**CHAOTIC DYNAMICS**

The self-oscillatory, homoclinic and excitable dynamics have been obtained both for resonant and no resonant excitation. We have also numerically observed chaotic evolution when the system operates in a detuned mode. As an example of the complexity of the system, and the requirement of detuned operation, in Fig. 4 (up) we show the stationary solutions, for different pump values, as a function of the detuning. Note that now two Hopf bifurcations are present for high drivings, defining a window where the solution is unstable. The stability analysis does not
predict the evolution in the unstable region (only the self-pulsed behavior near the bifurcations).

In order to explore the dynamical regimes in the unstable region, we have numerically integrated Eqs. (3) and represented the temporal spectrum as a function of detuning, for a fixed pump value. This corresponds to an experimental situation where the source frequency is varied, keeping constant the driving voltage. The obtained spectrogram is shown in Fig. 4(b).

FIGURE 4. (a) Stationary solutions depending on the pump, for different detuning values \( P=0.45, 1, 2 \) and 3.9). The symbols mark the Hopf bifurcations. Dashed lines correspond to unstable solutions. (b) Spectrum of the solutions, as the detuning is varied, for a pump value of \( P = 3.9 \).

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