Modelling of Acoustic Wave Propagation using Transient Insular Nodal Analysis (TINA)

PACS: 43.58.Ta

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ABSTRACT

A new time-domain method for computational acoustics is presented that yields magnitude and phase values for acoustic pressure and particle velocity at each point in the simulation grid. The approach is based on finite differences but, in contrast to classical finite-difference methods, is able to compute the system solution without the need for prediction, while maintaining fully-decoupled cells. Due to the use of transmission-line models for wave propagation, high computational efficiency can be achieved, while keeping the mathematical complexity low. Material boundary conditions do not need to be explicitly defined, but are taken automatically into account by the transmission-line links that make up the simulation grid. The result is a novel, unconditionally stable computation that can be arbitrarily scaled on modern grid computer clusters, as only direct-neighbour connectivity is required. The software program designed to implement the method allows a system to be modelled by drawing the physical structures and assigning material properties to them. A two-dimensional acoustic simulation is demonstrated, and compared with measurements, with good agreement.

INTRODUCTION

Traditional volume-discretisation methods suffer from a number of numerical drawbacks that limit their application in the study of practical acoustic systems. These issues are largely related to the discretisation rules used, as well as the resulting size and structure of the system matrices.

Numerically stable integration rules, such as backward Euler and trapezoidal, yield a non-diagonal system matrix. With an optimised mesh generator, the matrix will be banded, so sparsity methods can be used to solve it. However, it still requires the numerically costly process of matrix inversion. Given that the system matrices are large for a practical system, the allowable size is severely limited, due to excessive computation time. Parallelisation is an option, but does not map well on current computer clusters, as each node needs to exchange data with a central machine that handles the system matrix. Bandwidth and communication overhead quickly become a limiting factor in the scalability of such methods, as all communications must pass through this central machine.

One way around this limitation is the use of predictive integration methods, such as forward Euler, that yield a decoupled, diagonal system matrix. Such a system is efficient and maps well on modern grid clusters, as only communication with the neighbours is required. However, there are issues with the numerical stability of the integration rule, due to the use of prediction. In addition, regardless of the approach, the use of an integration rule results in non-exact discrete versions of the original PDEs.

The transient insular nodal analysis (TINA) method is able to model wave propagation without the need for an integration rule, or prediction. All computed time steps are solely based on previous ones, while still yielding a fully decoupled system. This results in efficient computations that scale well for parallel computing, and take full advantage of modern, multi-core CPUs.
BACKGROUND
Transient insular nodal analysis has its roots in transmission-line modelling as used for power-system studies. We will now touch on some of the key elements, as used in the TINA approach.

Electro-magnetic waves
Under many circumstances, electro-magnetic and acoustic waves can be described using the same form of wave equation. By solving Maxwell's equations [1], we can find a solution for a 1D plane wave parallel to the X-axis in a lossless, non-dispersive, homogeneous medium. For voltage \( E \) we find:

\[
\frac{\partial^2 E(x,t)}{\partial z^2} = \varepsilon \mu \frac{\partial^2 E(x,t)}{\partial t^2} \quad v = \frac{1}{\sqrt{\varepsilon \mu}} \quad Z_c = \sqrt{\frac{\mu}{\varepsilon}} \quad (Eq. 1)
\]

The two material parameters are the permittivity (\( \varepsilon \)) and permeability (\( \mu \)). They relate to the wave propagation speed (\( v \)) and characteristic impedance (\( Z_c \)) of the medium. Except for the medium properties, and the form of the energy, an identical equation can be obtained for an acoustic wave [1]. Voltage and current become pressure and particle velocity, while the permittivity and permeability of the medium become the density and compressibility of the material.

Constant-parameter transmission-line model
It is possible to obtain a discrete form of the solutions to the wave equation without discretisation. The resulting transmission-line model, described by Dommel [2,3] for use in the EMTP (Electro-Magnetic Transient Program, the UBC version of which is known as Microtran.) power-systems simulator is exact, efficient and based on the time-delay introduced by the wave propagating along the line. The CP (constant-parameter) model is lossless, but can be extended to allow losses, without additional computational costs.

The model is based upon the d'Alembert solutions to the wave equation (Eq. 1), and considers the voltages and currents at the terminals of the line (Figure 1). Since the line is initially lossless, it effectively represents a time delay.

\[
v(x,t) = v_f(x,t) + v_b(x,t) \quad i(x,t) = i_f(x,t) + i_b(x,t) \quad (Eq. 3)
\]

with \( f \) the forward wave, and \( b \) is the backward wave. The forward and backward currents (\( i_f \) and \( i_b \)) are related to the forward and backward voltages (\( v_f \) and \( v_b \)) by the characteristic impedance (\( Z_c \)). The relationship between the \( x \) and \( t \) arguments of the waves is defined by the velocity of propagation (\( a \)). Equation (3) can now be expressed as:
\[ v(x,t) = v_f(x-at) + v_b(x+at) \]

\[ i(x,t) = \frac{1}{Z_c} v_f(x-at) - \frac{1}{Z_c} v_b(x+at) \]  

(Eq. 4)

We can add both equations (Eq. 4), and write their total contribution as a forward wave for any point on the line (it suffers no reflection, regardless of termination):

\[ v(x,t) + Z_c i(x,t) = 2 v_f(x-at) \]  

(Eq. 5)

Defining the travel time \( \tau = \frac{l}{a} \), with \( l \) the line length, we can express the receiving end of the line as a function of the sending end, and find:

\[ v_m(t) + Z_c i_m(t) = v_k(t-\tau) + Z_c i_k(t-\tau) \]  

(Eq. 6)

where \( t \) is the present time and \( t - \tau \) the history terms. Re-formulated in a circuit equivalent (Figure 2) we find:

Figure 2.-CP line model

\[ e_{sb}(t) = v_m(t-\tau) + Z_c i_m(t-\tau) \]
\[ e_{mb}(t) = v_k(t-\tau) + Z_c i_k(t-\tau) \]  

(Eq. 7)

This model has many advantages over a direct solution of Maxwell's equations. It is not only numerically simple, but it also time-decouples nodes \( k \) and \( m \). The model is absolutely stable and exact (except for interpolation errors if \( \tau \) is not an integer multiple of the simulation time step \( \Delta t \)), as no integration rule was used in the derivation.

**TINA**

The fundamental problem with the way traditional volume-discretisation methods describe the system is that simultaneity is implicitly assumed in the description. The main concept behind the TINA method is that energy must travel at a finite speed. This concept, which lies at the origin of special and general relativity, invalidates the principle of simultaneity and re-enforces causality.

In practice, this means that all parts of the simulation, all discrete elements that make-up the mesh, are time-isolated from each other. One cannot, forbidden by fundamental laws of physics, have an immediate effect on spatially different elements in the system. As a result of this time-isolation, all spatially separated, discretised elements in the mesh are little 'islands' in time, which only need to take their direct neighbours into account, as more distant elements have yet to make their contributions known, and to do so have to propagate their energy through the elements in-between, which takes time.

The mathematical result is that only diagonal system matrices are allowed. Any off-diagonal non-zero values describe instantaneous, spatially separated influences which violate the physical wave-propagation-speed limitations of the medium.

**Wave propagation in transient simulations**

Many wave phenomena can be described as some form of energy, a perturbation, being propagated through a medium, and are governed by analogous equations. A physical model's behaviour, then, is solely determined by the nature and geometry of the medium through which the waves propagate.
When we wish to investigate the functioning of a structure in TINA, we assign properties to each physical part that makes up the system and set the, usually zero, initial conditions. We then present a forcing function to the whole. This forcing function will propagate through the medium segments and be subjected to transmission, refraction, and attenuation. Eventually, the system will reach steady-state. This way, we can use a wave-propagation model in a transient step-by-step simulation to study both the transient and steady-state responses of a structural model to an excitation. Function rises from structure.

The TINA grid
Our implementation of TINA is in the form of a time-domain, time-marching, finite-difference method. The system is composed of a regular grid of transmission lines (Figure 3), joined with nodes. We then assign medium properties to each cell.

Each cell has a central node, for which we compute the state variables, and between two and six (depending upon the dimensionality of the grid) line models connected to it. The particular line model used depends on the nature of the medium. It has to be noted that the use of a discrete medium results not only in time, but also spatial, discretisation. As such, we have to take two Nyquist conditions into account: one for the simulation time step, and one for the grid size.

Since the CP line model yields time-decoupling, a full nodal representation of the system is not required, as we can solve each node in the system independent of the others. As a result, each node is a parallel circuit of the respective halves of the CP line models corresponding to the physical propagation directions in the model. The other halves of the lines are time decoupled, and thus have no influence on the node in question during the time step. The node solution is obtained from Kirchoff’s current and voltage laws, which are an expression of conservation of energy. By analogy, voltage is acoustic pressure, and current becomes acoustic particle velocity.

Boundary conditions
In order to make our finite grid appear infinite in extent, the grid boundaries are terminated in self-adjusting loads. These cells automatically present the characteristic impedance of the line segments, and thus assure perfect matching. This approach, known as the "Perfectly Matched Layer" [5], prevents any spurious effects from the grid boundaries.

Other boundary conditions, eg. material interfaces, are implicit to the model. From transmission-line theory, when a discontinuity is encountered, part of the energy wave will be reflected back to the source, while part will be transmitted through the boundary. As such, we do not need to treat boundaries in any special way.
RESULTS
Due to the present limitations of the current code base, only a two-dimensional simulation could be performed. In order mimic this in experiment, we had to constrain the acoustic waves to obtain two-dimensional, plane-wave propagation. In this mode, the wave front is cylindrical, and follows a $1/\sqrt{r}$ pressure attenuation rule, as opposed to the usual $1/r$ for spherical wave fronts.

Experimental set-up
The system was built out of plywood sheets, spaced 3.8 cm apart using small wooden blocks, as illustrated in Figure 4. This physical distance allowed for the placement of the microphone and point source in-between. However, the distance also puts an effective upper limit on the frequency for which the two-dimensional approximation is valid. Furthermore, the spatial extent of the plywood was 120 cm around the source. With an acoustic wave speed of 345 m/s, a 3.8 cm wave guide has a cut-off frequency of about 4500 Hz. Thus 4000 Hz is a reasonable upper limit to the two-dimensional approximation. For the 120 cm dimension, we find 140 Hz. Hence, a 200 Hz lower bound was obtained.

Comparison with measurement
The transfer functions between point A and various other points in the system were computed from the raw measurement and simulation data. To reduce noise, the measurements were averaged over 16 acquisitions. Also, the simulation was done with an infinite-plane approximation, while the measurements have impedance discontinuities at the boundaries that could not yet be modelled, as this requires three-dimensional geometry to describe the end of the wooden panels. In addition, the simulation boundary still exhibits some reflection, due to a matching problem. These reflections cause significant distortion in the simulation output, which is clearly visible in the phase plot.

In order to make a meaningful comparison, both transfer functions were averaged to a single magnitude (Table 1) value between 200 Hz and 4000 Hz for the various positions indicated in Figure 4. Also, for the C-location, the magnitude and phase responses were over-layed for direct comparison (Figure 5 Left). The measurements agree reasonably well with the simulations, except for a constant factor. We attribute this to the boundary reflections.

CONCLUSIONS
The TINA method, although still in early development, shows promise for the modelling of real acoustic systems. It yields both magnitude and phase information while only requiring geometry and material properties as input, and is inherently stable.

Future work will focus on better boundary absorbers, expansion to three dimensions, code performance optimizations, and parallelisation. The modelling of losses and non-linearities will be investigated in addition to the use of controllers and active feed-back systems, such as active noise control.
Table I.- Averaged results for acoustic pressure

<table>
<thead>
<tr>
<th>Point</th>
<th>Pressure Measurement</th>
<th>Pressure Simulation</th>
<th>Error factor (mean = 0.77)</th>
<th>Deviation from mean error</th>
</tr>
</thead>
<tbody>
<tr>
<td>B → A</td>
<td>0.897</td>
<td>0.724</td>
<td>0.81</td>
<td>0.04</td>
</tr>
<tr>
<td>C → A</td>
<td>0.797</td>
<td>0.591</td>
<td>0.74</td>
<td>-0.03</td>
</tr>
<tr>
<td>D → A</td>
<td>0.692</td>
<td>0.514</td>
<td>0.74</td>
<td>-0.02</td>
</tr>
<tr>
<td>E → A</td>
<td>0.538</td>
<td>0.461</td>
<td>0.75</td>
<td>-0.02</td>
</tr>
<tr>
<td>F → A</td>
<td>0.540</td>
<td>0.421</td>
<td>0.78</td>
<td>0.01</td>
</tr>
<tr>
<td>G → A</td>
<td>0.492</td>
<td>0.390</td>
<td>0.79</td>
<td>0.03</td>
</tr>
<tr>
<td>H → A</td>
<td>0.488</td>
<td>0.362</td>
<td>0.74</td>
<td>-0.03</td>
</tr>
<tr>
<td>I → A</td>
<td>0.433</td>
<td>0.340</td>
<td>0.79</td>
<td>0.02</td>
</tr>
<tr>
<td>J → A</td>
<td>0.430</td>
<td>0.326</td>
<td>0.76</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

Figure 5.-Comparison of measured and simulated acoustic pressure (Point C → A)

References: