A GENERAL PROCEDURE FOR SELECTING SUITABLE EQUIVALENT MONOPOLE SETS FOR RIGID BODY SCATTERING PROBLEMS

PACS: 43.20.Fn, 43.20.El

Gounot (1), Yves J. R.; Musafir (1, 2), Ricardo E.
1, Acoustics and Vibration Lab./Dept. Mechanical Engineering/COPPE/UFRJ
2, Water Resources and Environmental Engineering Department/EP/UFRJ
Universidade Federal do Rio de Janeiro, Brazil;
ygounot@mecanica.ufrj.br; rem@serv.com.ufrj.br

ABSTRACT The main difficulty that one faces when using the Equivalent Sources Method for solving acoustic scattering or radiation problems is to pick an adequate number of sources and choose their appropriate positioning. In a previous work, the authors proposed rules leading to suitable and easy-to-implement monopole arrangements for simple geometry scatterers like parallelepipeds. In this paper, it is proposed a simple procedure to determine appropriate source arrangements for scatterers with a more complex geometry, guaranteeing, still with a low number of monopoles, a satisfactory response. Solutions are compared to the ones obtained with the expansions in spherical wave functions technique and the procedure effectiveness is explained in terms of the resulting set of the monopole source strengths.

INTRODUCTION

Because of its low computational cost, the equivalent sources method (ESM) is an interesting alternative to boundary elements method (BEM), being able to yield an approximate solution with a small number of sources. It substitutes the real body by a set of sources located in its interior chosen in order to satisfy the appropriate boundary condition [1-2]. Although in the case of expansions, the issue of the source positioning can be considered solved [1], choosing the number and the position of the equivalent sources still constitutes an important difficulty when sets of monopoles are used since, in this case, the solution quality depends strongly on these variables [3]. In spite of some results and recommendations relative to some particular cases, no general rules that could constitute working guidelines for the user are available. Scattering from relatively simple geometry structures will, frequently, present a high degree of symmetry, so that it is reasonable to assume the existence of simple a priori appropriate source arrangements. After recalling some early but fundamental results relative to easy-to-implement efficient source supports for parallelepipedic-shaped scatterers [4], this paper presents a simple and general procedure that allows dealing with more complex scatterers geometry, what should allow a straightforward and safer use of the ESM.

THEORETICAL BACKGROUND AND EARLY RESULTS

The scattering problem and the equivalent source method

The exterior scattering problem due to the impinging of an acoustic wave on a body can be described, in the frequency domain, as follows [5]: the complex scattered pressure \( p_{sc}(x, \omega) \) has to satisfy both Helmholtz equation and Sommerfeld radiation condition in the exterior domain and, on the body boundary \( S \), the prescribed normal velocity, given, for rigid bodies, by

\[
\bar{u}_n = -v_{inc}^n, \quad \text{where} \quad v_{inc}^n = \text{normal velocity that would be generated by the incident wave in the absence of the body.}
\]

\( ESM \) substitutes the real body by a set of \( M \) sources at points \( y_m \), located strictly inside the region previously filled by the body volume, \( V \). The pressure and velocity fields due to these sources are expressed in terms of their unknown complex amplitudes (the source strengths \( A_m \)) and a function \( g(x, y_m) \) describing their radiation,

\[
p_{sc}(x) = \sum_{m=1}^{M} A_m \ g(x, y_m) \quad v_{inc}^n(x) = \frac{-1}{\imath \omega \rho_0} \sum_{m=1}^{M} A_m \ \frac{\partial g(x, y_m)}{\partial n} \quad \text{(Eq. 1-2)}
\]
where $\omega = k c_0$ is the angular frequency, $k$ is the wave number, $c_0$ is the speed of sound and $\rho_0$ is the exterior domain uniform mean density. The function $g$ most commonly used is the free-space Green function and expansions in spherical wave functions (which correspond to multipoles) [6]. The solution — the source strength set $\{A_m\}$ — is obtained by minimizing the velocity error on the boundary, i.e., the difference between the normal velocity generated by the source set on $S$ and the corresponding prescribed values.

**Numerical experiment general description**

It is well-known that the quality of the solution depends on the number of monopoles employed and on their positioning inside the body. A previous study involving parallelepiped-shaped scatterers [4] has shown some typical features of appropriate monopole arrangements, which lead up to rules and guidelines responding to the recurrent question: how many sources should be used and how should their positioning be chosen. The main results, presented below, are relative to the scattering of a plane wave by a rigid parallelepiped with dimension $(\eta \times 1 \times 1) \lambda$, its largest dimension being referred as $L_\eta$. Different cases are investigated, depending on the value of the scatterer aspect ratio $\eta$ ($\eta = 1, 2, 3$ and $4$) and also on the wave incidence, given by the orientation of the wave vector $k$ (see Fig. 1).

![Figure 1.- Representation of the scatterer with the source supports used for the two incidence cases](image)

**Figure 1.- Representation of the scatterer with the source supports used for the two incidence cases**

Different source supports of easy computational implementation were used: a linear one parallel to $k$ (L), circular (C), elliptical (E) and a double linear one (LL) — made of two parallel lines normal to $k$, containing, each, $M/2$ monopoles. All are located in the $z = 0$ plane, which is a symmetry plane for the scattered field. Their centers are coincident with the body geometric center and their sizes are obtained by multiplying the maximum size acceptable (such that all sources are strictly inside the body) by a reduction factor $a$, $0 < a < 1$. The source configurations are referred as $L_M$, $C_M$, $E_M$ and $LL_M$ where $M$ denotes the total number of monopoles regularly positioned on the support. For each case investigated (i.e., for a given body, frequency, support type and size), 24 solutions are computed, corresponding to $M = 2$ to 25 monopoles. The solution quality is evaluated through the normalized quadratic boundary velocity error $e_{BC}$.

\[
e_{BC} = \frac{\sum_{i=1}^{N} \left| v_{in}(x_i) - \bar{u}_n(x_i) \right|^2}{\left( \sum_{i=1}^{N} \left| \bar{u}_n(x_i) \right|^2 \right)}.
\] (Eq. 3)

As for the quality criterion, numerous trials relative to the case of a cubic scatterer (with side equal to the wavelength $\lambda$) allowed to determine a limiting upper value of the acceptable $e_{BC}$ as 0.6. This value has been chosen for corresponding to a good pressure field reconstitution in a control circle with radius $2\lambda$ (i.e., reasonably close to the boundary), an error lower than 1 dB being obtained for 95% of the control points.

**Previous results**

For the cube case ($\eta = 1$), it has been shown that the linear support always permits better solutions than the circular one. Moreover, when $e_{BC}$ values are comparable, the number of monopoles required with L is always significantly smaller than with C. The linear support efficiency is explained in terms of the typical source strength distribution obtained: the monopoles are basically forming a sequence of dipoles; as an illustration, the source magnitude and phase corresponding to the $L_2$ solution for $kL = 2\pi$, are shown in Fig.2a. When the whole source set is considered, a high degree of cancellation is obtained between the in-line
monopoles. The emission occurs mainly along the support axis, allowing the reconstitution of
the corresponding directional scattered fields. Furthermore, it has been shown that the non-
dimensional values of the distance between two adjacent sources, \( kd_S \), corresponding to the
best solutions are nearly constant. This result yields a rule relating the cube edge and the
frequency to the ‘optimal’ number of monopoles as,

\[ M \approx 1 + 0.66 k L. \quad \text{(Eq. 4)} \]

It was verified that Eq. 4 provides, if not exactly the number of monopoles associated to the best
solution, numbers that always correspond to a solution fulfilling the quality criterion adopted.

For scatterers presenting an aspect ratio higher than unity, two notable situations were
considered: \( k \) parallel to and \( k \) normal to the largest side of the body \( (L_\eta) \). When \( k \) is parallel to
\( L_\eta \), for all \( \eta \) tested the only appropriate support that furnishes good quality response with a small
number of monopoles is the linear one and the rule given for the cube (Eq. 4) is still valid,
provided \( L \) is substituted by \( L_\eta \). When \( k \) is normal to \( L_\eta \), as \( \eta \) increases, the only suitable ‘single’
support is the elliptical one, since both linear and circular ones failed.

On the other hand, it has been shown that the double linear support is efficient in all cases,
notably for the highest \( \eta \) value used. Furthermore, compared to the best ‘single’ configurations
obtained (E), the number of monopoles necessary with LL to obtain a good solution is reduced
by a factor of 4. The typical source arrangement obtained with the double linear support is
illustrated in Figure 2b, showing the formation of an array of parallel dipoles (made of the
monopoles numbered as ‘1’ and ‘2’, ‘3’ and ‘4’, etc…). A study on the influence, on the solution
quality, of the number of pairs of monopoles and of their relative positioning along the scatterer
symmetry axis, has shown that the most economical source arrangement that guarantees a good
solution with this support is given by

\[ M \approx 2(\eta + 1) \quad \text{and} \quad a^* \approx 0.17\left(\frac{1}{2} M - 1\right). \quad \text{(Eq. 5-6)} \]

These rules permit, for paralelepipedic-shaped scatterers, determining easy-to-implement
appropriate monopole arrangements.

**CASE OF SCATTERERS WITH A MORE COMPLEX GEOMETRY**
In this section, a simple and general procedure for dealing with more complex scatterer geometry is proposed, which is based on approximating the structures by a finite number of elements with parallelepipedic shape. Each one of these substructures is to be ‘substituted’ by a monopole-source arrangement, determined using the rules given in the previous section. Then, the problem is solved globally, considering, as ‘the’ source set, the union of these individual source sets. In order to guarantee that the collection of sources still represents an adequate equivalent source set for the whole body, a special attention must be paid to the manner the structure breaking up is done, this determinant point being illustrated in the following example. A 3D ‘L-shaped’ structure is considered (see Fig. 3): its largest dimensions are \((2 \times 3 \times 1) \lambda\) and its faces are referred to as \(F_1, \ldots, F_8\); \(F_7\) and \(F_8\) are the lower and upper faces, i.e., those in the \(z = -\lambda/2\) and \(z = \lambda/2\) planes, respectively.

Figure 3.- Representation of the L-shaped structure and the two simplest breaking-up cases with the corresponding appropriate supports.

This structure geometry is such that the two simplest ways to break it up into two elements, \(e_1\) and \(e_2\), are:

- case nº1: two parallelepipeds with dimensions \((1 \times 2 \times 1) \lambda\) and \((2 \times 1 \times 1) \lambda\);
- case nº2: one parallelepiped \((1 \times 3 \times 1) \lambda\) plus one cube of side \(\lambda\).

These two cases yield two different combinations of source sets: in the first case, \(LL_6 \oplus L_9\) for \(e_1\) and \(e_2\), respectively, and \(LL_8 \oplus L_5\) in the second one. Table I shows, for these two situations, the boundary error corresponding to the whole structure \(e_{BC}\) as well as the contribution (in percentage) of each one of its eight faces for the total error. In order to evaluate the efficiency of these monopole arrangements, comparison is made with the solution obtained using expansions according to a procedure proposed by Ochmann [1]. It consists in dividing the body in a set of cubic or spherical substructures, in whose centers are placed the expansion points. Here, expansions with order up to \(\alpha\), referred as \(X^\alpha\), are used in the geometric center of the 4 cubes in which the structure is divided. It is worth underlying the fact that \(X^\alpha\) contains \((\alpha+1)^2\) functions, i.e., an expansion referred to as \(X^1, X^2, X^3\) or \(X^4\) corresponds to 4, 9, 16 or 25 ‘sources’ with different orders, respectively: monopole and dipoles for \(\alpha = 1\), monopole, dipoles and quadrupoles for \(\alpha = 2\), etc.

**TABLE I.** The \(e_{BC}\) values corresponding to the two breaking up cases of the L-shaped structure; ‘Ochmann expansions’ \(e_{BC}\) results.

<table>
<thead>
<tr>
<th>Configurations</th>
<th>(M) (total)</th>
<th>(e_{BC})</th>
<th>% contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(LL_6 \oplus L_9)</td>
<td>15</td>
<td>(0.50)</td>
<td>(24) (6) (7) (8) (20) (3) (16) (16)</td>
</tr>
<tr>
<td>Case nº2</td>
<td>13</td>
<td>(1.03)</td>
<td>(60) (3) (3) (15) (0) (8) (8)</td>
</tr>
<tr>
<td>‘expansion technique’</td>
<td>(X^1)</td>
<td>16</td>
<td>(0.39)</td>
</tr>
<tr>
<td></td>
<td>(X^2)</td>
<td>36</td>
<td>(0.32)</td>
</tr>
<tr>
<td></td>
<td>(X^3)</td>
<td>64</td>
<td>(0.18)</td>
</tr>
<tr>
<td></td>
<td>(X^4)</td>
<td>100</td>
<td>(0.17)</td>
</tr>
</tbody>
</table>

Results show that, in the first case, the source set \((LL_6 \oplus L_9)\) yields a solution with a pretty good precision \((e_{BC} = 0.5)\) for a quite low number of monopoles \((M = 15)\), about 20 times smaller than the number of nodes (288). It was verified that the solution corresponding to the
proposed rules, besides yielding a satisfactory $e_{BC}$, always presents a good homogeneity in the precision over the entire structure boundary.

As for the second case, the solution obtained using LL$_8$ coupled with L$_5$ shows, relatively to the first case one, a significant loss in precision, the corresponding $e_{BC}$ being about twice higher. This $e_{BC}$ deterioration is essentially due to the poor velocity reconstitution on F$_1$ (which, in this case, concentrates 60% of the error — see Table I) and F$_5$, as evidenced in Figure 4, which shows the normal velocity generated by the different source sets considered.

![Figure 4. - Normal velocity at the 'L-shaped structure' boundary nodes and corresponding theoretical values with the expansions (left) and for the two breaking up cases.](image)

The 'expansions' results show that, globally, the solution quality increases with the order of the expansions: the reconstructed normal velocity is getting closer and closer to both maximum values (on F$_1$, F$_3$ and F$_5$) and minimum ones (on the other faces). The fact the expansions allow a good velocity distribution in the 3D space is obviously an advantage over the one layer monopole sets. However, the expansion superior efficiency can be very costly, since the first configuration yielding an solution accuracy significantly better than the one obtained with the 15-monopoles set (case n°1) involves 4 expansions up to order 3, i.e., containing 64 sources, the higher order ones corresponding to octupoles.

As for the solution quality obtained with the two different monopole sets, the discrepancy can be explained as follows. The 'a priori' appropriate source configurations (the linear and the double linear one) have been determined separately for the individual sub-structures. When the sub-structures are connected, the nodes located on the now common surface, B$_{12}$, which correspond to inner points of the 'new' structure, do not exist anymore (see Fig.3). On the other hand, since the driving nodes for the determination of the source strengths are those in which the normal velocity is non-zero, the missing nodes (the ones on B$_{12}$) have a much more important contribution to the boundary condition in the second case than in the first one. As a consequence, while in the first case, the union of the two sub-source sets still corresponds to an adequate set when the whole L-shaped structure is considered, this does not happen in the second one, in which the sources located close to B$_{12}$ (in the two sub-sets) are sort of ‘confused’ by the absence of the important boundary part. This phenomenon can be seen in Fig. 5, which shows the source strengths obtained in the 2 cases. While the LL sub-source sets still present the feature of formation of dipoles in both cases, this does not happen.
with the ‘single’ linear sub-source set L: while for the first decomposition case, L₉ still shows a sequence of monopoles basically in phase opposition as in the ‘ideal’ linear arrangement, for the second case, the phase alternancy is no more obtained with L₅. The loss of the intrinsic structure of the linear sub-set seems to be responsible for the efficiency loss observed in the boundary condition reconstitution.

CONCLUDING REMARKS
A general procedure furnishing a suitable monopole set for the equivalent source method applied to scattering problems has been proposed. It consists in substituting the structure by a proper number of parallelepipeds and then, endowing an appropriated ‘ready-to-use’ monopole set to each one of the sub-structure. The way the structure is divided into parallelepipeds is therefore a crucial point to guarantee that the individual appropriate monopole sets will still constitute, together, an appropriate source set for the complete structure. The simplest way to express the adequate procedure for the structure division is that the separating virtual surface between two substructures should be taken parallel to the wave vector. This procedure constitutes a helpful working guideline that should contribute for an easier and safer use of the equivalent source method.

Acknowledgement: Financial support was provided by the National Research Council of Brazil, CNPq.

References: